

REPRESENTATIONS OF QUIVER HECKE ALGEBRAS VIA LYNDON BASES

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ABSTRACT. A new class of algebras has been introduced by Khovanov and Lauda and independently by Rouquier. These algebras categorify one-half of the Quantum group associated to arbitrary Cartan data. In this paper, we use the combinatorics of Lyndon words to construct the irreducible representations of those algebras associated to Cartan data of finite type. This completes the classification of simple modules for the quiver Hecke algebra initiated by Kleshchev and Ram.

1. INTRODUCTION

1.1. Recently, Khovanov and Lauda [KL1, KL2] and Rouquier [Rq] have independently introduced a remarkable family of graded algebras, $H(\Gamma)$, defined in terms of quivers associated to the Dynkin diagram, Γ , of a symmetrizable Kac-Moody algebra, \mathfrak{g} . These algebras have been given several names, including Khovanov-Lauda-Rouquier algebras, quiver nil-Hecke algebras, quiver Hecke algebras, and “the rings $R(\nu)$ ” (here ν refers to an element in the positive cone \mathcal{Q}^+ inside the root lattice of \mathfrak{g}). The main property of these algebras is that

$$K(\Gamma) \cong \mathcal{U}_{\mathcal{A}}^*(\mathfrak{n})$$

as *twisted* bimodules, where $K(\Gamma)$ is the Grothendieck group of the full subcategory, $\text{Rep}(\Gamma)$, of finite dimensional graded $H(\Gamma)$ -modules, \mathfrak{n} is a maximal nilpotent subalgebra of \mathfrak{g} , and $\mathcal{U}_{\mathcal{A}}^*(\mathfrak{n})$ is an integral form of the quantized enveloping algebra, $\mathcal{U}_q(\mathfrak{n})$.

Further evidence of the importance of these algebras was obtained in [BK2]. In this work, Brundan and Kleshchev showed that when Γ is of type A_∞ or $A_{\ell-1}^{(1)}$, there is an isomorphism between blocks of cyclotomic Hecke algebras of symmetric groups, and blocks of a corresponding quotient of $H(\Gamma)$. Moreover, this isomorphism applies equally well to the Hecke algebra and its rational degeneration, depending only on Γ and the underlying ground field. In light of the work [BK1], it is expected that a similar relationship should hold between interesting quotients of $H(\Gamma)$ and cyclotomic Hecke-Clifford algebras when Γ is of type B_∞ and $A_{2\ell}^{(2)}$. For these reasons, we choose to use the name “*quiver Hecke algebra*” to describe $H(\Gamma)$.

1.2. In [HKS], Hill, Kujawa and Sussan investigated the representation theory of the (degenerate) affine Hecke-Clifford algebra, $\mathcal{HC}(d)$, over \mathbb{C} . In this paper, the authors constructed an analogue of the Arakawa-Suzuki functor [AS] between the category \mathcal{O} for the Lie superalgebra $\mathfrak{q}(n)$ and a certain category, $\text{Rep}\mathcal{HC}(d)$, of *integral* finite dimensional modules for $\mathcal{HC}(d)$. By considering small rank instances of the functor, the authors obtained analogues of Zelevinsky’s *segment representations*, [BZ, Z], for $\mathcal{HC}(d)$. More generally, the Verma modules for $\mathfrak{q}(n)$ correspond under the functor to certain induced modules, which by [HKS, Theorem 4.4.10] have unique irreducible quotients. The

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authors went on to obtain a construction of all the irreducible integral representations using the combinatorics of Lyndon words together with [BK1, Theorem 7.17].

It is instructive to describe their result in more detail. To this end, let Γ be a Dynkin diagram of finite type with nodes labelled by the index set I , fix a total ordering, \leq , on I . Let \mathcal{F} be the free associative algebra generated by the letters $[i]$, $i \in I$, with the concatenation product $[i_1] \cdots [i_k] = [i_1, \dots, i_k]$, and give the monomials in \mathcal{F} the lexicographic ordering determined by I . It was first noticed in [LR] that certain monomials in \mathcal{F} associated to this ordering, called *good Lyndon words*, and their non-increasing products, called *good words*, naturally determine various bases of the quantized enveloping algebra, $\mathcal{U}_q(\mathfrak{n})$, of a maximal nilpotent subalgebra of the semisimple Lie algebra \mathfrak{g} associated to Γ . This observation was further developed in the prophetic paper of Leclerc, [Le], where it was first suggested that the bases arising from these combinatorics should naturally correspond to representations of affine Hecke algebras, cf. [Le, Sections 6-7], specifically [Le, Theorem 47, Conjecture 52].

In [HKS], the authors noticed that the character of each segment representation of $\mathcal{HC}(d)$ corresponds in a natural way to a dual canonical basis element labeled by a good Lyndon word in type B with respect to the standard Dynkin ordering on I (specialized at $q = 1$), see [HKS, Proposition 4.1.3, Theorem 4.1.8, Proposition 8.2.12]. This was a nontrivial observation since it applied only after redeveloping the theory so that monomials are ordered lexicographically from *right-to-left*, a technicality imposed by the functor, cf. [HKS, Lemma 8.2.13]. This choice had the effect of drastically simplifying both the good Lyndon words in type B , and their associated dual canonical basis elements. More generally, the characters of standard modules naturally correspond to dual PBW basis elements labeled by good words (again, at $q = 1$), [HKS, Theorem 8.5.1]. Finally, applying [HKS, Theorem 4.4.10] completed the construction, [HKS, Theorem 8.5.5].

Motivated by the results of [HKS] and the conjectured connection between $\mathcal{HC}(d)$ and quiver Hecke algebras of type B , we initiated a study of the representation theory of the category $\text{Rep}(\Gamma)$, for Γ of classical finite type, using the combinatorics of Lyndon words with respect to the standard Dynkin ordering on I and the right-to-left lexicographic ordering described in [HKS]. Indeed, we first observed that this simplified the good Lyndon words in every type (except for the long roots in type C , which remain the same). Subsequently, we worked out the corresponding dual canonical basis elements, b_l^* , associated to each good Lyndon word, l , and constructed representations, $\mathbf{1}_l$, with character b_l^* . The standard module, $\mathcal{M}(g)$, associated to a good word g is the module obtained by parabolic induction:

$$\mathcal{M}(g) = \text{Ind } \mathbf{1}_{l_1} \boxtimes \cdots \boxtimes \mathbf{1}_{l_k} \{c_g\},$$

where $g = l_1 \cdots l_k$ is the canonical factorization of g as a non-increasing product of good Lyndon words and the term $\{c_g\}$ refers to a grading shift. These standard modules have the property that their characters are given by dual PBW basis elements labelled by the corresponding good word, and, therefore, give a basis for the Grothendieck group, $K(\Gamma)$.

1.3. While this paper was in production, Kleshchev and Ram completed their own investigation of $\text{Rep}(\Gamma)$ using the combinatorics of Lyndon words, for Γ of *arbitrary finite type*. To describe this paper in more detail, give I an arbitrary total ordering. The authors called an irreducible $H(\Gamma)$ -module *cuspidal* if its character is given by a dual canonical basis element associated to a

good Lyndon word, cf. [KR2, Lemma 6.4]. They went on to prove an amazing lemma. Namely, given a cuspidal representation, $\mathbf{1}_l$, the module

$$\mathcal{M}(l^k) = \text{Ind } \underbrace{\mathbf{1}_l \boxtimes \cdots \boxtimes \mathbf{1}_l}_{k \text{ times}} \{c_{l^k}\}$$

remains irreducible for all $k > 1$, [KR2, Lemma 6.6]. We want to point out that this lemma applies equally well to all possible orderings on I and all finite root systems. Combining [KR2, Lemma 6.6] with a straightforward Frobenius reciprocity argument shows that the standard module $\mathcal{M}(g)$ has a unique irreducible quotient $\mathcal{L}(g)$, [KR2, Theorem 7.2]. In this way, Kleshchev and Ram reduced the study of $\text{Rep}(\Gamma)$ to the construction of cuspidal representations. They went on to construct all cuspidal representations in types $ABCDG$ as well as E_6 and E_7 using the standard Dynkin ordering, the good Lyndon words in [LR, Le], and the corresponding root vectors in [Le, Section 8], cf. [KR2, Section 8]. In type A they produced cuspidal representations for all orderings on I .

1.4. Given the beautiful results in [KR2], we expanded the goal of this paper. In particular, our main result is a complete determination of the cuspidal representations of $H(\Gamma)$ in all finite types using our ordering, Theorem 4.1.1. We would like to point out several advantages of our approach. First, in classical type, our cuspidal representations tend to be much simpler than those appearing in [KR2]. More specifically, in types BCD , our representations generally have dimension at most 2 (with the exception of the long roots in type C). In contrast, the cuspidal modules constructed by Kleshchev and Ram generally have dimensions that grow with the height of an associated positive root. Another advantage can be seen when considering the case of E_8 . The main difficulty for Kleshchev and Ram is that not all the E_8 root vectors are homogeneous in the sense of [KR1], see [KR2, Section 8]. On the other hand, in our ordering, all good Lyndon words in type E_8 are homogeneous. Finally, in §2.5, 3.4, and 4.3 we explain exactly how to relate the right-to-left lexicographic ordering used here to the more standard left-to-right lexicographic ordering in [Le] and [KR2].

In this paper, we only use the half of the bialgebra structure of $K(\Gamma)$ coming from parabolic induction. It would also be interesting to consider the structure coming from restriction and compare the work here to that of Lauda and Vazirani, [LV].

Finally, we would like to point out that the description of the simple modules for the quiver Hecke algebra of type B is nearly identical to the description of the irreducible $\mathcal{HC}(d)$ -modules appearing in [HKS]. In particular, it is possible to define an action of (an appropriately defined) *quiver Hecke-Clifford* superalgebra of type B on the segment representations of $\mathcal{HC}(d)$. Moreover, this action extends easily to standard modules. Based on small rank calculations, we conjecture that this action factors through the unique simple quotients. We feel that an investigation of this phenomenon should shed light into the relationship between the type B quiver Hecke algebra and the Hecke-Clifford algebra, but this is a topic of another paper.

1.5. The remainder of the paper is organized as follows. In Section 2 we describe the embedding of the quantum group $\mathcal{U}_q(\mathfrak{n})$ inside the q -shuffle algebra \mathcal{F} and describe the combinatorics of Lyndon words in our set-up following [Le] and [HKS, Section 8] closely. In Section 3 we introduce the quiver Hecke algebra and describe some of the basic properties of the category $\text{Rep}(\Gamma)$. In Section 4 we

introduce cuspidal representations, and standard representations and state the main theorem of the paper, Theorem 4.1.1. In Section 5 we determine the good Lyndon words and corresponding root vectors, and Section 6 contains the construction of cuspidal representations. Finally, Appendix A contains the calculations relevant to Section 5.

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2. QUANTUM GROUPS

2.1. Root Data. Let \mathfrak{g} be a simple finite dimensional Lie algebra of rank r over \mathbb{C} , with Dynkin diagram Γ and let I denote the set of labels of the nodes of Γ . Let $\mathcal{U}_q(\mathfrak{g})$ be the corresponding quantum group over $\mathbb{Q}(q)$ with Chevalley generators e_i, f_i , $i \in I$. Let $\mathfrak{n} \subseteq \mathfrak{g}$ be the subalgebra generated by the e_i , $i \in I$. Let Δ be the root system of \mathfrak{g} relative to this decomposition, Δ^+ the positive roots, and $\Pi = \{\alpha_i | i \in I\}$ the simple roots. Let \mathcal{Q} be the root lattice and $\mathcal{Q}^+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$. Let $A = (a_{ij})_{i,j \in I}$ be the Cartan matrix of \mathfrak{g} and (\cdot, \cdot) denote symmetric bilinear form on \mathfrak{h}^* satisfying

$$a_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}, \quad d_i = \frac{(\alpha_i, \alpha_i)}{2} \in \{1, 2, 3\}.$$

Let $q_i = q^{d_i}$. Define the q -integers and q -binomial coefficients:

$$[k]_i = \frac{q_i^k - q_i^{-k}}{q_i - q_i^{-1}}, \quad [k]_i! = [k]_i \cdots [2]_i [1]_i, \quad \begin{bmatrix} m \\ k \end{bmatrix}_i = \frac{[m]_i!}{[k]_i! [m-k]_i!}.$$

For later purposes, we also define the following. Let $\nu \in \mathcal{Q}^+$, say $\nu = \sum_{i \in I} c_i \alpha_i$. Define the *height* of ν :

$$\text{ht}(\nu) = \sum_{i \in I} c_i.$$

Next, given $\underline{i} = (i_1, \dots, i_d) \in I^d$, define the *content* of \underline{i} by

$$\text{cont}(\underline{i}) = \sum_{i \in I} n_i \alpha_i, \quad n_i = \#\{j = 1, \dots, d \mid i_j = i\}.$$

Finally, if $\text{ht}(\nu) = d$, set $I^\nu = \{\underline{i} \in I^d \mid \text{cont}(\underline{i}) = \nu\}$. Let S_d denote the symmetric group on d letters, generated by simple transpositions s_1, \dots, s_{d-1} . Then, S_d acts I^d by place permutation and we denote this action by $w \cdot \underline{i}$, $w \in S_d$, $\underline{i} \in I^d$. Observe that the orbits of this action are precisely the sets I^ν with $\text{ht}(\nu) = d$.

2.2. Embedding of $\mathcal{U}_q(\mathfrak{n})$ in the Quantum Shuffle Algebra. The algebra $\mathcal{U}_q := \mathcal{U}_q(\mathfrak{n})$ is a quotient of the free algebra generated by the Chevalley generators e_i , $i \in I$ by the relations

$$\sum_{r+s=1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_i e_i^r e_j e_i^s = 0.$$

It is naturally \mathcal{Q}^+ -graded by assigning to e_i the degree α_i . Let $|u|$ be the \mathcal{Q}^+ -degree of a homogeneous element $u \in \mathcal{U}_q$.

In [K], Kashiwara proved that there exist q -derivations e'_i , $i \in I$ given by

$$e'_i(e_j) = \delta_{ij} \quad \text{and} \quad e'_i(uv) = e'_i(u)v + q^{-(\alpha_i, |u|)} u e'_i(v)$$

for all homogeneous $u, v \in \mathcal{U}_q$. For each $i \in I$, $e'_i(u) = 0$ if, and only if $|u| = 0$.

Now, let \mathcal{F} be the free associative algebra over $\mathbb{Q}(q)$ generated by the set of letters $\{[i] \mid i \in I\}$. Letters should not be confused with q -integers, which always occur with a subscript. Write $[i_1, \dots, i_k] := [i_1] \cdot [i_2] \cdots [i_k]$, and let $[]$ denote the empty word. The algebra \mathcal{F} is \mathbb{Q}^+ graded by assigning the degree α_i to $[i]$ (as before, let $|f|$ denote the \mathbb{Q}^+ -degree of a homogeneous $f \in \mathcal{F}$). Notice that \mathcal{F} also has a *principal grading* obtained by setting the degree of a letter $[i]$ to be 1; let \mathcal{F}_d be the d th graded component in this grading.

Now, define the (quantum) shuffle product, $*$, on \mathcal{F} inductively by

$$(x \cdot [i]) * (y \cdot [j]) = (x * (y \cdot [j])) \cdot [i] + q^{-(|x| + \alpha_i, \alpha_j)} ((x \cdot [i]) * y) \cdot [j], \quad x * [] = [] * x = x. \quad (2.2.1)$$

Iterating this formula yields

$$[i_1, \dots, i_\ell] * [i_{\ell+1}, \dots, i_{\ell+k}] = \sum_{w \in D_{(\ell, k)}} q^{-e(w)} [i_{w^{-1}(1)}, \dots, i_{w^{-1}(\ell+k)}] \quad (2.2.2)$$

where $D_{(\ell, k)}$ is the set of minimal coset representatives in $S_{\ell+k}/S_\ell \times S_k$ and

$$e(w) = \sum_{\substack{s \leq \ell < t \\ w(s) < w(t)}} (\alpha_{i_s}, \alpha_{i_t}),$$

see [Le, §2.5]. The product $*$ is associative and, [Le, Proposition 1],

$$x * y = q^{-(|x|, |y|)} y \bar{x} \quad (2.2.3)$$

where \bar{x} is obtained by replacing q with q^{-1} in the definition of $*$.

Now, to $f = [i_1, \dots, i_k] \in \mathcal{F}$, associate $\partial_f = e'_{i_1} \cdots e'_{i_k} \in \text{End} \mathcal{U}_q$, and $\partial_{[]} = \text{Id}_{\mathcal{U}_q}$. Then,

Proposition 2.2.1. [R1, R2, G] *There exists an injective $\mathbb{Q}(q)$ -linear homomorphism*

$$\Psi : \mathcal{U}_q \rightarrow (\mathcal{F}, *)$$

defined on homogeneous $u \in \mathcal{U}_q$ by the formula $\Psi(u) = \sum \partial_f(u)f$, where the sum is over all monomials $f \in \mathcal{F}$ such that $|f| = |u|$.

Therefore \mathcal{U}_q is isomorphic to the subalgebra $\mathcal{W} \subseteq (\mathcal{F}, *)$ generated by the letters $[i]$, $i \in I$.

Let $\mathcal{A} = \mathbb{Q}[q, q^{-1}]$, and let $\mathcal{U}_{\mathcal{A}}$ denote the \mathcal{A} -subalgebra of \mathcal{U}_q generated by the divided powers $e_i^k/[k]_i!$ ($i \in I$, $k \in \mathbb{Z}_{\geq 0}$). Let $(\cdot, \cdot)_K : \mathcal{U}_q \times \mathcal{U}_q \rightarrow \mathbb{Q}(q)$ denote the unique symmetric bilinear form satisfying

$$(1, 1)_K = 1 \quad \text{and} \quad (e'_i(u), v)_K = (u, e_i v)_K \quad (2.2.4)$$

for all $i \in I$, and $u, v \in \mathcal{U}_q$. Let

$$\mathcal{U}_{\mathcal{A}}^* = \{ u \in \mathcal{U}_q \mid (u, v)_K \in \mathcal{A} \text{ for all } v \in \mathcal{U}_{\mathcal{A}} \} \quad (2.2.5)$$

and let $u^* \in \mathcal{U}_{\mathcal{A}}^*$ denote the dual to $u \in \mathcal{U}_{\mathcal{A}}$ relative to $(\cdot, \cdot)_K$. It is well known that for $u \in \mathcal{U}_{\mathcal{A}, \nu}$, the map $u^* \mapsto (u^*, ?)_K$ defines an isomorphism $\mathcal{U}_{\mathcal{A}, \nu}^* \cong \text{Hom}_{\mathcal{A}}(\mathcal{U}_{\mathcal{A}, \nu}, \mathcal{A})$.

Remark 2.2.2. Observe that the form we are using differs slightly from Lustig's bilinear form $(\cdot, \cdot)_L$. They are related by the formula

$$(u, v)_L = \prod_{i \in I} \frac{1}{(1 - q_i^2)^{c_i}} (u, v)_K,$$

if $|u| = |v| = \sum_i c_i \alpha_i$. In particular, if B is a basis of \mathcal{U}_q consisting of homogeneous vectors, then the adjoint basis of B with respect to $(\cdot, \cdot)_K$ and $(\cdot, \cdot)_L$ differ only by some normalization factors. In particular, B is orthogonal with respect to $(\cdot, \cdot)_K$ if, and only if it is orthogonal with respect to $(\cdot, \cdot)_L$.

Throughout this paper, we will follow Leclerc and use the form $(\cdot, \cdot)_K$. In §3.5 we will explain how both forms arise in representation theory, cf. Example 3.5.5 and Lemma 3.5.6.

Now, given a monomial

$$[i_1^{a_1}, i_2^{a_2}, \dots, i_k^{a_k}] = [\underbrace{i_1, \dots, i_1}_{a_1}, \underbrace{i_2, \dots, i_2}_{a_2}, \dots, \underbrace{i_k, \dots, i_k}_{a_k}]$$

with $i_j \neq i_{j+1}$ for $1 \leq j < k$, let $c_{i_1, \dots, i_k}^{a_1, \dots, a_k} = [a_1]_{i_1}! \cdots [a_k]_{i_k}!$, so that $(c_{i_1, \dots, i_k}^{a_1, \dots, a_k})^{-1} e_{i_1}^{a_1} \cdots e_{i_k}^{a_k}$ is a product of divided powers. Let

$$\mathcal{F}_{\mathcal{A}}^* = \bigoplus \mathcal{A} c_{i_1, \dots, i_k}^{a_1, \dots, a_k} [i_1^{a_1}, i_2^{a_2}, \dots, i_k^{a_k}]$$

and $\mathcal{W}_{\mathcal{A}}^* = \mathcal{W} \cap \mathcal{F}_{\mathcal{A}}^*$. It is known that $\mathcal{W}_{\mathcal{A}}^* = \Psi(\mathcal{U}_{\mathcal{A}}^*)$, [Le, Lemma 8].

We close this section by describing some simple involutions of \mathcal{F} which correspond, on restriction to \mathcal{W} , to important involutions on \mathcal{U}_q . To this end, for $\nu = \sum_i c_i \alpha_i \in \mathcal{Q}^+$, define

$$N(\nu) = \frac{1}{2} \left((\nu, \nu) - \sum_{i=1}^r c_i (\alpha_i, \alpha_i) \right). \quad (2.2.6)$$

Proposition 2.2.3. [Le, Proposition 6] *Let $f = [i_1, \dots, i_k]$, $|f| = \nu$. Then,*

(i) *Let $\tau : \mathcal{F} \rightarrow \mathcal{F}$ be the $\mathbb{Q}(q)$ -linear map defined by $\tau(f) = [i_k, \dots, i_1]$. Then, $\tau(x * y) = \tau(y) * \tau(x)$ for all $x, y \in \mathcal{F}$. Hence, $\tau(\Psi(u)) = \Psi(\tau(u))$, where $\tau : \mathcal{U}_q \rightarrow \mathcal{U}_q$ is the $\mathbb{Q}(q)$ -linear anti-automorphism which fixes the generators e_i .*

(ii) *Let $- : \mathcal{F} \rightarrow \mathcal{F}$ be the \mathbb{Q} -linear map defined by $\bar{q} = q^{-1}$ and*

$$\bar{f} = q^{N(\nu)} [i_k, \dots, i_1].$$

*Then, $\overline{x * y} = \bar{x} * \bar{y}$ for all $x, y \in \mathcal{F}$. Hence, $\overline{\Psi(u)} = \Psi(\bar{u})$, where $-$ is the bar involution on \mathcal{U}_q .*

(iii) *Let $\sigma : \mathcal{F} \rightarrow \mathcal{F}$ be the \mathbb{Q} -linear map such that $\sigma(q) = q^{-1}$ and*

$$\sigma(f) = q^{N(\nu)} f.$$

Then, $\sigma(x) = \overline{\tau(x)}$ for all $x \in \mathcal{F}$. Hence, $\Psi(\sigma(u)) = \sigma(\Psi(u))$, where $\sigma : \mathcal{U}_q \rightarrow \mathcal{U}_q$ is the \mathbb{Q} -linear anti-automorphism which sends q to q^{-1} and fixes the Chevalley generators e_i .

2.3. Good Words and Lyndon Words. In what follows, our conventions differ from those in [Le]. In particular, we order monomials in \mathcal{F} lexicographically reading from *right to left*. Except for the type A case, this convention leads to some significant differences in the good Lyndon words that appear. For the convenience of the reader, we include §2.5 which explains the connection between the combinatorics developed using this ordering to those which arise using the more common *left to right* lexicographic ordering.

The next two sections parallel [Le, Sections 3,4] with the statements of the relevant propositions adjusted to conform to our choice of ordering.

For the remainder of the section, fix an ordering on the set of letters $\{[i] | i \in I\}$ in \mathcal{F} , denoted \leq , and order Π accordingly. Give the set of monomials in \mathcal{F} the associated lexicographic order read from right to left, also denoted \leq . That is, set $[i] < []$ for all $i \in I$ and

$$[i_1, \dots, i_k] < [j_1, \dots, j_\ell] \text{ if } i_k < j_\ell, \text{ or for some } m, i_{k-m} < j_{\ell-m} \text{ and } i_{k-s} = j_{\ell-s} \text{ for all } s < m.$$

Note that since the empty word is larger than any letter, every word is smaller than all of its right factors:

$$[i_1, \dots, i_k] < [i_j, \dots, i_k], \text{ for all } 1 < j \leq k. \quad (2.3.1)$$

(For those familiar with the theory, this definition is needed to ensure that the induced Lyndon ordering on positive roots is convex, cf. §2.4 below.)

For a homogeneous element $f \in \mathcal{F}$, let $\min(f)$ be the smallest monomial occurring in the expansion of f . A monomial $[i_1, \dots, i_k]$ is called a *lower good word* if there exists a homogeneous $w \in \mathcal{W}$ such that $[i_1, \dots, i_k] = \min(w)$, and we say that it is *Lyndon on the right* if it is larger than any of its proper left factors:

$$[i_1, \dots, i_j] < [i_1, \dots, i_k], \text{ for any } 1 \leq j < k.$$

Except for §2.5, we refer to these special words simply as *good* and *Lyndon*. Let \mathcal{G} denote the set of good words, \mathcal{L} the set of Lyndon words, and $\mathcal{GL} = \mathcal{L} \cap \mathcal{G} \subset \mathcal{G}$ the set of good Lyndon words. Also, let $\mathcal{GL}_d \subset \mathcal{G}_d \subset \mathcal{F}_d$ denote the degree d components of \mathcal{GL} and \mathcal{G} in the principal grading. Finally, for $\nu \in \mathcal{Q}^+$, let $\mathcal{GL}_\nu \subset \mathcal{G}_\nu \subset \mathcal{F}_\nu$ be the homogeneous components of \mathcal{GL} and \mathcal{G} in the \mathcal{Q}^+ grading.

Lemma 2.3.1. [Le, Lemma 13] *Every factor of a good word is good.*

Because of our ordering conventions, [Le, Lemma 15, Proposition 16] become

Lemma 2.3.2. [Le, Lemma 15] *Let $l \in \mathcal{L}$, w a monomial such that $w \geq l$. Then, $\min(w * l) = wl$.*

and

Proposition 2.3.3. [Le, Proposition 16] *Let $l \in \mathcal{GL}$, and $g \in \mathcal{G}$ with $g \geq l$. Then $gl \in \mathcal{G}$.*

Hence, we deduce from Lemma 2.3.1 and Proposition 2.3.3 [Le, Proposition 17]:

Proposition 2.3.4. [LR, Le] *A monomial g is a good word if, and only if, there exist good Lyndon words $l_1 \geq \dots \geq l_k$ such that*

$$g = l_1 l_2 \cdots l_k.$$

As in [Le], we have

Proposition 2.3.5. [LR, Le] *The map $l \rightarrow |l|$ is a bijection $\mathcal{GL} \rightarrow \Delta^+$.*

Given $\beta \in \Delta^+$, let $\beta \rightarrow l(\beta)$ be the inverse of the above bijection (called the Lyndon covering of Δ^+).

We now define the *bracketing* of Lyndon words, that gives rise to the *Lyndon basis* of \mathcal{W} . To this end, given $l \in \mathcal{L}$ such that l is not a letter, define the standard factorization of l to be $l = l_1 l_2$ where $l_2 \in \mathcal{L}$ is a proper left factor of maximal length. Define the q -bracket

$$[f_1, f_2]_q = f_1 f_2 - q^{(|f_1|, |f_2|)} f_2 f_1 \quad (2.3.2)$$

for homogeneous $f_1, f_2 \in \mathcal{F}$ in the \mathcal{Q}^+ -grading. Then, the bracketing $\langle l \rangle$ of $l \in \mathcal{L}$ is defined inductively by $\langle l \rangle = l$ if l is a letter, and

$$\langle l \rangle = [\langle l_1 \rangle, \langle l_2 \rangle]_q \quad (2.3.3)$$

if $l = l_1 l_2$ is the standard factorization of l .

Example 2.3.6. For \mathfrak{g} of type B_r with I given in Table 5.1 below, we have

- (1) $\langle [0] \rangle = [0]$;
- (2) $\langle [12] \rangle = [[1], [2]]_q = [12] - q^{-2}[21]$;
- (3) $\langle [012] \rangle = [[0], [12] - q^{-2}[21]]_q = [012] - q^{-2}[021] - q^{-2}[120] + q^{-4}[210]$.

As is suggested in this example, we have

Proposition 2.3.7. [Le, Proposition 19] *For $l \in \mathcal{L}$, $\langle l \rangle = l + r$ where r is a linear combination of words w such that $|w| = |l|$ and $w < l$.*

Any word $w \in \mathcal{F}$ has a canonical factorization $w = l_1 \cdots l_k$ such that $l_1, \dots, l_k \in \mathcal{L}$ and $l_1 \geq \cdots \geq l_k$. We define the bracketing of an arbitrary word w in terms of this factorization: $\langle w \rangle = \langle l_1 \rangle \cdots \langle l_k \rangle$. Define a homomorphism $\Xi : (\mathcal{F}, \cdot) \rightarrow (\mathcal{F}, *)$ by $\Xi([i]) = [i]$. Then, $\Xi([i_1, \dots, i_k]) = [i_1] * \cdots * [i_k] = \Psi(e_{i_1} \cdots e_{i_k})$. In particular, $\Xi(\mathcal{F}) = \mathcal{W}$. We have the following characterization of good words:

Lemma 2.3.8. [Le, Lemma 21] *The word w is good if and only if it cannot be expressed modulo $\ker \Xi$ as a linear combination of words $v < w$.*

For $g \in \mathcal{G}$, set $r_g = \Xi(\langle g \rangle)$. Then, we have

Theorem 2.3.9. [Le, Proposition 22, Theorem 23] *Let $g \in \mathcal{G}$ and $g = l_1 \cdots l_k$ be the canonical factorization of g as a nonincreasing product of good Lyndon words. Then*

- (1) $r_g = r_{l_1} * \cdots * r_{l_k}$,
- (2) $r_g = \Psi(e_g) + \sum_{w < g} x_{gw} \Psi(e_w)$ where, for a word $v = [i_1, \dots, i_k]$, $e_v = e_{i_1} \cdots e_{i_k}$, and
- (3) $\{r_g | g \in \mathcal{G}\}$ is a basis for \mathcal{W} .

The basis $\{r_g | g \in \mathcal{G}\}$ is called the Lyndon basis of \mathcal{W} . An immediate consequence of Proposition 2.3.7 and Theorem 2.3.9 is the following:

Proposition 2.3.10. [Le, Proposition 24] *Assume $\beta_1, \beta_2 \in \Delta^+$, $\beta_1 + \beta_1 = \beta \in \Delta^+$, and $l(\beta_1) < l(\beta_2)$. Then, $l(\beta_1)l(\beta_2) \geq l(\beta)$.*

This gives an inductive algorithm to determine $l(\beta)$ for $\beta \in \Delta^+$ (cf. [Le, §4.3]):

For $\alpha_i \in \Pi \subset \Delta^+$, $l(\alpha_i) = [i]$. If β is not a simple root, then there exists a factorization $l(\beta) = l_1 l_2$ with l_1, l_2 Lyndon words. By Lemma 2.3.1, l_1 and l_2 are good, so $l_1 = l(\beta_1)$ and $l_2 = l(\beta_2)$ for some $\beta_1, \beta_2 \in \Delta^+$ with $\beta_1 + \beta_2 = \beta$. Assume that we know $l(\beta_0)$ for all $\beta_0 \in \Delta^+$ satisfying $\text{ht}(\beta_0) < \text{ht}(\beta)$. Define

$$C(\beta) = \{ (\beta_1, \beta_2) \in \Delta^+ \times \Delta^+ \mid \beta = \beta_1 + \beta_2, \text{ and } l(\beta_1) < l(\beta_2) \}.$$

Then, Proposition 2.3.10 implies

Proposition 2.3.11. [Le, Proposition 25] *We have*

$$l(\beta) = \min\{ l(\beta_1)l(\beta_2) \mid (\beta_1, \beta_2) \in C(\beta) \}$$

2.4. PBW and Canonical Bases. The lexicographic ordering on \mathcal{GL} induces a total ordering on Δ^+ , which is *convex*, meaning that if $\beta_1, \beta_2 \in \Delta^+$ with $\beta_1 < \beta_2$, and $\beta = \beta_1 + \beta_2 \in \Delta^+$, then $\beta_1 < \beta < \beta_2$ (cf. [R3, Le]). Indeed, assume $\beta_1, \beta_2, \beta = \beta_1 + \beta_2 \in \Delta^+$ and $\beta_1 < \beta_2$. Proposition 2.3.10 and (2.3.1) imply that $l(\beta) \leq l(\beta_1)l(\beta_2) < l(\beta_2)$. If $l(\beta) = l(\beta_1)l(\beta_2)$, then the definition of Lyndon words implies $l(\beta_1) < l(\beta)$. We are therefore left to prove that $l(\beta_1) < l(\beta)$ even if $l(\beta) < l(\beta_1)l(\beta_2)$. This can be checked easily in all cases. We call this ordering a (right) Lyndon ordering on Δ^+ .

Now, [Le, Corollary 27] becomes

Corollary 2.4.1. *Let $\beta \in \Delta^+$. Then, $l(\beta)$ is the largest good word of weight β .*

Each convex ordering, $\beta_1 < \dots < \beta_N$, on Δ^+ arises from a unique decomposition $w_0 = s_{i_1} s_{i_2} \dots s_{i_N}$ of the longest element of the Weyl group of \mathfrak{g} via

$$\beta_1 = \alpha_{i_1}, \beta_2 = s_{i_1} \alpha_{i_2}, \dots, \beta_N = s_{i_1} \dots s_{i_{N-1}} \alpha_{i_N}.$$

Lusztig associates to this data a PBW basis of $\mathcal{U}_{\mathcal{A}}$ denoted

$$E^{(a_1)}(\beta_1) \dots E^{(a_N)}(\beta_N), \quad (a_1, \dots, a_N) \in \mathbb{Z}_{\geq 0}^N.$$

Leclerc [Le, §4.5] describes the image in \mathcal{W} of this basis for the convex Lyndon ordering. We use the same braid group action as Leclerc and the results of [Le, §4.5, 4.6] carry over, making changes in the same manner indicated in the previous section. We describe the relevant facts below.

For $g = l(\beta_1)^{a_1} \dots l(\beta_k)^{a_k}$, where $\beta_1 > \dots > \beta_k$ and $a_1, \dots, a_k \in \mathbb{Z}_{>0}$ set

$$E_g = \Psi(E^{(a_k)}(\beta_k) \dots E^{(a_1)}(\beta_1)) \in \mathcal{W}_{\mathcal{A}}$$

and let $E_g^* \in \mathcal{W}_{\mathcal{A}}^*$ be the image of $(E^{(a_k)}(\beta_k) \dots E^{(a_1)}(\beta_1))^* \in \mathcal{U}_{\mathcal{A}}^*$. Observe that the order of the factors in the definition of E_g above are increasing with respect to the Lyndon ordering. Leclerc shows that if $\beta \in \Delta^+$, then

$$\kappa_{l(\beta)} E_{l(\beta)} = r_{l(\beta)}, \tag{2.4.1}$$

For some $\kappa_{l(\beta)} \in \mathbb{Q}(q)$, [Le, Theorem 28] (the proof of this theorem in our case is obtained by reversing all the inequalities and using the standard factorization as opposed to the costandard

factorization). More generally, let $g = l_1^{a_1} \cdots l_k^{a_k} \in \mathcal{G}$, $l_1 > \cdots > l_k \in \mathcal{GL}$. If $l = l(\beta)$, write $d_l := d_i$ if $(\beta, \beta) = (\alpha_i, \alpha_i)$, and define

$$\kappa_g = \prod_{i=1}^k \kappa_{l_i}^{a_i} [a_i]_{l_i}!. \quad (2.4.2)$$

Then, $E_g = \kappa_g \sigma(r_g)$, where σ is defined in Proposition 2.2.3, [Le, §4.6]. Moreover,

$$E_g^* = q^{c_g} (E_{l_m}^*)^{*a_m} * \cdots * (E_{l_1}^*)^{*a_1} \quad (2.4.3)$$

where $c_g = \sum_{i=1}^m \binom{a_i}{2} d_{l_i}$, [Le, §5.5.3].

It is well known that using the bar involution (Proposition 2.2.3) we obtain a canonical basis $\{b_g \mid g \in \mathcal{G}\}$ for $\mathcal{W}_{\mathcal{A}}$ via the PBW basis $\{E_g \mid g \in \mathcal{G}\}$, see [Le, Lemma 37]. It has the form

$$b_g = E_g + \sum_{\substack{h \in \mathcal{G} \\ h < g}} \chi_{gh} E_h. \quad (2.4.4)$$

The dual canonical basis then has the form

$$b_g^* = E_g^* + \sum_{\substack{h \in \mathcal{G} \\ h > g}} \chi_{gh}^* E_h^*. \quad (2.4.5)$$

As in [Le] we have the following very important theorem:

Theorem 2.4.2. [Le, Theorem 40, Corollary 41]

- (i) We have $\min(b_g^*) = g$ for all $g \in \mathcal{G}$. Moreover, the coefficient of g in b_g^* is equal to κ_g .
- (ii) For each $l \in \mathcal{GL}$, $E_l^* = b_l^*$.

To describe the coefficient κ_l precisely, transport the symmetric bilinear form (2.2.4) to \mathcal{W} via the isomorphism Ψ . Let $g = l(\beta_1)^{a_1} \cdots l(\beta_N)^{a_N}$ and $h = l(\beta_1)^{b_1} \cdots l(\beta_N)^{b_N}$, where $a_1, \dots, a_N, b_1, \dots, b_N \in \mathbb{Z}_{\geq 0}$. Then, the form is given by

$$(E_g, E_h)_K = \delta_{gh} \prod_{j=1}^n \frac{(E(\beta_j), E(\beta_j))_K^{a_j}}{\{a_j\}_{(\beta_j, \beta_j)}!} \quad (2.4.6)$$

where, for $\beta = \sum_{i=1}^r c_i \alpha_i \in \Delta^+$,

$$(E(\beta), E(\beta))_K = \frac{\prod_{i=1}^r (1 - q^{(\alpha_i, \alpha_i)})^{c_i}}{1 - q^{(\beta, \beta)}} \quad (2.4.7)$$

and for $a, b \in \mathbb{Z}_{\geq 0}$,

$$\{a\}_b! = \prod_{j=1}^a \frac{1 - q^{jb}}{1 - q^b}. \quad (2.4.8)$$

Then, [Le, §5.5.2],

$$E_l^* = \frac{(-1)^{\ell(l)-1} \kappa_l^{-1}}{q^{N(|l|)} (E_l, E_l)_K} r_l, \quad (2.4.9)$$

where $N(|l|)$ is given by (2.2.6)

2.5. The Anti-Automorphism τ . We continue with a fixed ordering, \leq , on I and corresponding sets \mathcal{G} , \mathcal{L} , and \mathcal{GL} as described in §2.3. Define the *opposite* ordering on I by

$$x \preceq y \text{ if, and only if, } y \leq x.$$

Given this opposite ordering, define the corresponding opposite total ordering on the monomials in \mathcal{F} by

$$[i_1, \dots, i_k] \prec [j_1, \dots, j_\ell] \text{ if } i_1 \prec j_1, \text{ or for some } m, i_m \prec j_m \text{ and } i_s = j_s \text{ for all } s < m,$$

and $[] \prec [i]$ for all $i \in I$.

For $f \in \mathcal{F}$, $\max(f)$ is the largest monomial occuring in the expansion of f . Call a monomial $g^\tau = [i_1, \dots, i_k] \in \mathcal{F}$ an *upper good word* if $g^\tau = \max(u)$ for some $u \in \mathcal{U}_q$, and we say that it is *Lyndon on the left* if it is smaller than all of its proper right factors:

$$[i_1, \dots, i_k] \prec [i_j, \dots, i_k] \text{ for } j > 1.$$

Let \mathcal{G}^τ denote the set of upper good words, let \mathcal{L}^τ denote the set of words that are Lyndon on the left, and $\mathcal{GL}^\tau = \mathcal{G}^\tau \cap \mathcal{L}^\tau$.

Observe that the total ordering on \mathcal{GL}^τ induces a convex total ordering on Δ^+ which we call a (left) Lyndon ordering. Also, the bijection $\Delta^+ \rightarrow \mathcal{GL}^\tau$ provides a means to compute $l^\tau(\beta)$ for each $\beta \in \Delta^+$, see [Le, Section 4]. Finally, given $l^\tau \in \mathcal{L}^\tau$, define its costandard factorization to be $l^\tau = l_1^\tau l_2^\tau$, where l_1^τ is the maximal proper word which is Lyndon on the left. Note that l_2^τ is also Lyndon on the left. Using the data above we may define a Lyndon basis $\{r_{g^\tau} | g^\tau \in \mathcal{G}^\tau\}$, dual PBW basis $\{E_{g^\tau}^* | g^\tau \in \mathcal{G}^\tau\}$ and dual canonical basis $\{b_{g^\tau}^* | g^\tau \in \mathcal{G}^\tau\}$ exactly as in [Le, Sections 4-5].

The next lemma gives the precise connection between the combinatorics appearing here and those developed in [Le]:

Lemma 2.5.1. *Under the anti-automorphism $\tau : \mathcal{F} \rightarrow \mathcal{F}$,*

- (1) $\tau(\mathcal{W}) = \mathcal{W}$;
- (2) $\tau(\mathcal{G}) = \mathcal{G}^\tau$ and $\tau(\mathcal{L}) = \mathcal{L}^\tau$;
- (3) $\tau(E_g^*) = E_{\tau(g)}^*$;
- (4) $\tau(b_g^*) = b_{\tau(g)}^*$.

Proof: Property (1) is immediate from Proposition 2.2.3, and property (2) is clear from the definitions.

We now turn to property (3). Observe that if $g = l_1 \cdots l_k$, then $\tau(g) = \tau(l_k) \cdots \tau(l_1)$. Therefore, by equation (2.4.3), it is enough to show that $\tau(E_l^*) = E_{\tau(l)}^*$ for all $l \in \mathcal{GL}$. We prove this by induction on the degree of l in the principal grading on \mathcal{F} . The base case is clear since $E_{[i]}^* = r_{[i]} = [i]$.

For the inductive step, assume we have shown that $\tau(r_{l_0}) = r_{\tau(l_0)}$ and $\tau(E_{l_0}^*) = E_{\tau(l_0)}^*$ for all l_0 of degree less than the degree of l . Let $l = l_1 l_2$ be the standard factorization of l . Then, by (2), $\tau(l) = \tau(l_2) \tau(l_1)$ is the costandard factorization of $\tau(l)$. Then, it follows from (2.3.3) and the

relevant definitions that

$$\begin{aligned}
\tau(r_l) &= \tau(r_{l_1} * r_{l_2} - q^{(|l_1|, |l_2|)} r_{l_2} * r_{l_1}) \\
&= \tau(r_{l_2}) * \tau(r_{l_1}) - q^{(|l_1|, |l_2|)} \tau(r_{l_1}) * \tau(r_{l_2}) \\
&= r_{\tau(l_2)} * r_{\tau(l_1)} - q^{(|l_2|, |l_1|)} r_{\tau(l_1)} * r_{\tau(l_2)} \\
&= r_{\tau(l)}.
\end{aligned}$$

It now follows that $\tau(E_l^*) = E_{\tau(l)}^*$ by applying τ to equation (2.4.9) and observing that equations (2.2.6) and (2.4.6)-(2.4.8) imply that the coefficient on the right-hand-side of (2.4.9) depend only on $|l| \in \mathcal{Q}^+$.

Finally, property (4) for follows by applying τ to equation (2.4.5) and uniqueness. ■

From now on, we will write $g^\tau = \tau(g)$.

3. QUIVER HECKE ALGEBRAS

In this section, we give a presentation of the quiver Hecke algebras following the notation of [KR2]. Throughout, we work over an arbitrary ground field \mathbb{F} .

3.1. Quivers with Compatible Automorphism. Let $\tilde{\Gamma}$ be a graph. We construct a Dynkin diagram Γ by giving $\tilde{\Gamma}$ the structure of a *graph with compatible automorphism* in the sense of [L, §12, 14]. To define the quiver Hecke algebra, we will use the notion of a *quiver with compatible automorphism* as described in [Rq, §3.2.4].

Let \tilde{I} be the labelling set for $\tilde{\Gamma}$, and \tilde{H} be the (multi)set of edges. An automorphism $a : \tilde{\Gamma} \rightarrow \tilde{\Gamma}$ is said to be *compatible* with $\tilde{\Gamma}$ if, whenever $(i, j) \in \tilde{H}$ is an edge, i is not in the orbit of j under a .

Fix a compatible automorphism $a : \tilde{\Gamma} \rightarrow \tilde{\Gamma}$, and set I to be a set of representatives of the orbits of \tilde{I} under a and, for each $i \in I$, let $\alpha_i \in \tilde{I}/a$ be the corresponding orbit. For $i, j \in I$, $i \neq j$ define $(\alpha_i, \alpha_j) = 2|\alpha_i|$ and let

$$(\alpha_i, \alpha_j) = -|\{(i', j') \in \tilde{H} \mid i' \in \alpha_i, j' \in \alpha_j\}|.$$

For all $i, j \in I$, let $a_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$. Then, [L, Proposition 14.1.2] $A = (a_{ij})_{i, j \in I}$ is a Cartan matrix and every Cartan matrix arises in this way. Let Γ be the Dynkin diagram corresponding to A . Moreover, the pairing (α_i, α_j) defined above agrees with the pairing on \mathcal{Q} in §2.1.

Assume further that $\tilde{\Gamma}$ is a quiver. That is, we have a pair of maps $s : \tilde{H} \rightarrow \tilde{I}$ and $t : \tilde{H} \rightarrow \tilde{I}$ (the source and the target). We say that a is a compatible automorphism if $s(a(h)) = a(s(h))$ and $t(a(h)) = a(t(h))$ for all $h \in \tilde{H}$. Set

$$d_{ij} = |\{h \in \tilde{H} \mid s(h) \in \alpha_i \text{ and } t(h) \in \alpha_j\}|/a|$$

and let $m(i, j) = \text{lcm}\{(\alpha_i, \alpha_i), (\alpha_j, \alpha_j)\}$. As noted in [Rq],

$$d_{ij} + d_{ji} = -2(\alpha_i, \alpha_j)/m(i, j). \quad (3.1.1)$$

This data defines a matrix $Q = (Q_{ij}(u, v))_{i, j \in I}$, where each $Q_{ij}(u, v) \in \mathbb{F}[u, v]$. The polynomial entries in Q are defined by $Q_{ii}(u, v) = 0$, and for $i \neq j$,

$$Q_{ij}(u, v) = (-1)^{d_{ij}} (u^{m(i, j)/(\alpha_i, \alpha_i)} - v^{m(i, j)/(\alpha_j, \alpha_j)})^{-2(\alpha_i, \alpha_j)/m(i, j)} \quad (3.1.2)$$

Specialize now to the case where Γ is of finite type. Then, as explained in [KR2, §3.1], the polynomials $Q_{ij}(u, v)$ ($i \neq j$) are completely determined by the Cartan matrix and a partial ordering on I such that $i \rightarrow j$ or $j \rightarrow i$ if $a_{ij} \neq 0$. In this case,

$$Q_{ij}(u, v) = \begin{cases} 0 & \text{if } i = j; \\ 1 & \text{if } a_{ij} = 0; \\ u^{-a_{ij}} - v^{-a_{ji}} & \text{if } a_{ij} < 0 \text{ and } i \rightarrow j; \\ v^{-a_{ji}} - u^{-a_{ij}} & \text{if } a_{ij} < 0 \text{ and } j \rightarrow i. \end{cases} \quad (3.1.3)$$

3.2. Generators and Relations. Assume from now on that \mathfrak{g} is as in §2.1. Define the quiver Hecke algebra

$$H(\Gamma) = \bigoplus_{\nu \in \mathbb{Q}^+} H(\Gamma; \nu),$$

where $H(\Gamma; \nu)$ is the unital \mathbb{F} -algebra, with identity 1_ν , given by generators and relations as described below.

Assume that $\text{ht}(\nu) = d$. The set of generators are

$$\{e(\underline{i}) | \underline{i} \in I^\nu\} \cup \{y_1, \dots, y_d\} \cup \{\phi_1, \dots, \phi_{d-1}\}.$$

We refer to the $e(\underline{i})$ as *idempotents*, the y_r as *Jucys-Murphy elements*, and the ϕ_r as *intertwining elements*. Indeed, these generators are subject to the following relations for all $\underline{i}, \underline{j} \in I^\nu$ and all admissible r, s :

$$e(\underline{i})e(\underline{j}) = \delta_{\underline{i}, \underline{j}}e(\underline{i}); \quad (3.2.1)$$

$$\sum_{\underline{i} \in I^\nu} e(\underline{i}) = 1_\nu; \quad (3.2.2)$$

$$y_r e(\underline{i}) = e(\underline{i}) y_r; \quad (3.2.3)$$

$$\phi_r e(\underline{i}) = e(s_r \cdot \underline{i}) \phi_r; \quad (3.2.4)$$

$$y_r y_s = y_s y_r; \quad (3.2.5)$$

$$\phi_r y_s = y_s \phi_r \quad \text{if } s \neq r, r+1; \quad (3.2.6)$$

$$\phi_r \phi_s = \phi_s \phi_r \quad \text{if } |s - r| > 1; \quad (3.2.7)$$

$$\phi_r y_{r+1} e(\underline{i}) = \begin{cases} (y_r \phi_r + 1) e(\underline{i}) & i_r = i_{r+1}, \\ y_r \phi_r e(\underline{i}) & i_r \neq i_{r+1}; \end{cases} \quad (3.2.8)$$

$$y_{r+1} \phi_r e(\underline{i}) = \begin{cases} (\phi_r y_r + 1) e(\underline{i}) & i_r = i_{r+1}, \\ \phi_r y_r e(\underline{i}) & i_r \neq i_{r+1}. \end{cases} \quad (3.2.9)$$

Additionally, the intertwining elements satisfy the quadratic relations

$$\phi_r^2 e(\underline{i}) = Q_{i_r, i_{r+1}}(y_r, y_{r+1}) e(\underline{i}) \quad (3.2.10)$$

for all $0 \leq r < d$, and the braid-like relations

$$\begin{aligned} & (\phi_r \phi_{r+1} \phi_r - \phi_{r+1} \phi_r \phi_{r+1}) e(\underline{i}) \\ &= \begin{cases} \left(\frac{Q_{i_r, i_{r+1}}(y_{r+2}, y_{r+1}) - Q_{i_r, i_{r+1}}(y_r, y_{r+1})}{y_{r+2} - y_r} \right) e(\underline{i}) & \text{if } i_r = i_{r+2}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \quad (3.2.11)$$

Finally, this algebra is graded via

$$\deg e(\underline{i}) = 0, \quad \deg y_r e(\underline{i}) = (\alpha_{i_r}, \alpha_{i_r}), \quad \text{and} \quad \deg \phi_r e(\underline{i}) = -(\alpha_{i_r}, \alpha_{i_{r+1}}). \quad (3.2.12)$$

3.3. Basis Theorem. Let $\nu \in \mathcal{Q}^+$ with $\text{ht}(\nu) = d$. Given $w \in S_d$, fix a reduced decomposition $w = s_{k_1} \cdots s_{k_t}$ for w and define

$$\phi_w = \phi_{k_1} \cdots \phi_{k_t}.$$

Relations (3.2.10) and (3.2.11) imply that, in general, ϕ_w depends on the choice of reduced decomposition.

Finally, we have

Theorem 3.3.1. [KL1, Theorem 2.5][Rq, Theorem 3.7] *The set*

$$\{ \phi_w y_1^{m_1} \cdots y_d^{m_d} e(\underline{i}) \mid w \in S_d, m_1, \dots, m_d \in \mathbb{Z}_{\geq 0}, \underline{i} \in I^\nu \}$$

forms an \mathbb{F} -basis for $H(\Gamma; \nu)$.

3.4. An Automorphism and Anti-Automorphism of $H(\Gamma; \nu)$. Let $\nu \in \mathcal{Q}^+$, $\text{ht}(\nu) = d$. As observed in [KL1, §2.1], we have the following

Proposition 3.4.1. *There is a unique \mathbb{F} -linear automorphism $\tau : H(\Gamma; \nu) \rightarrow H(\Gamma; \nu)$ given by $\tau(e(i_1, \dots, i_d)) = e(i_d, \dots, i_1)$, $\tau(y_r) = y_{d-r+1}$, and $\tau(\phi_r) = -\phi_{d-r}$.*

and

Proposition 3.4.2. *There is a unique \mathbb{F} -linear anti-automorphism $\psi : H(\Gamma; \nu) \rightarrow H(\Gamma; \nu)$ defined by $\psi(e(\underline{i})) = e(\underline{i})$, $\psi(y_r) = y_r$, and $\psi(\phi_r) = \phi_r$ for all $\underline{i} \in I^\nu$ and admissible r .*

3.5. Modules and Graded Characters. Given a finite dimensional \mathbb{Z} -graded vector space $V = \bigoplus_{k \in \mathbb{Z}} V[k]$, define the *graded dimension* of V to be

$$\dim_q V = \sum_{k \in \mathbb{Z}} (\dim V[k]) q^k \in \mathbb{Z}_{\geq 0}[q, q^{-1}].$$

Let $V\{s\}$ denote the vector space obtained from V by shifting the grading by s . That is,

$$\dim_q V\{s\} = q^s \dim_q V.$$

The algebra $H(\Gamma; \nu)$ is \mathbb{Z} -graded by (3.2.12). Let $\text{Rep}(\Gamma; \nu)$ denote the category of all finite dimensional graded $H(\Gamma; \nu)$ -modules. Let M be in $\text{Rep}(\Gamma; \nu)$. For each $\underline{i} \in I^\nu$, define the generalized \underline{i} -eigenspace by $M_{\underline{i}} := e(\underline{i})M$. We have the decomposition

$$M = \bigoplus_{\underline{i} \in I^\nu} M_{\underline{i}}.$$

Moreover, by (3.2.4), $\phi_r M_{\underline{i}} = M_{s_r \cdot \underline{i}}$. Finally, note that since the elements $y_r e(\underline{i})$ have positive degree, they act nilpotently on all objects in $\text{Rep}(\Gamma; \nu)$.

Morphisms are *degree 0* $H(\Gamma; \nu)$ -homomorphisms. That is, for each $M, N \in \text{Rep}(\Gamma; \nu)$, $\text{hom}_\nu(M, N)$ denotes the set of *degree 0* homomorphisms.

Let $K(\Gamma; \nu) = K(\text{Rep}(\Gamma; \nu))$ be the Grothendieck group of the category $\text{Rep}(\Gamma; \nu)$, and

$$K(\text{Rep}(\Gamma)) := K(\Gamma) = \bigoplus_{\nu \in \mathcal{Q}^+} K(\Gamma; \nu).$$

This is a free $\mathbb{Z}[q, q^{-1}]$ -module with basis given by isomorphism classes of simple $H(\Gamma)$ -modules. Note that since morphisms have degree 0, $L \not\cong L\{s\}$ for any simple module $L \in \text{Rep}(\Gamma; \nu)$ and any $s \neq 0$. We write $[M] \in K(\Gamma; \nu)$ for the image of $M \in \text{Rep}(\Gamma; \nu)$ in the Grothendieck group. Finally, observe that $q^s[M] = [M\{s\}]$. Define the *formal character* $\text{ch} : \text{Rep}(\Gamma; \nu) \rightarrow \mathcal{F}$ by

$$\text{ch } M = \sum_{\underline{i} \in I^\nu} (\dim_q M_{\underline{i}}) \cdot [\underline{i}].$$

Theorem 3.5.1. [KL1, Theorem 3.17] *The character map induces an injective $\mathbb{Q}(q)$ -linear map*

$$\text{ch} : K(\Gamma; \nu) \rightarrow \mathcal{F}.$$

Now, let $\nu, \nu' \in \mathcal{Q}^+$ and let $H(\Gamma; \nu, \nu') := H(\Gamma; \nu) \otimes H(\Gamma; \nu')$. Given $\underline{i} = (i_1, \dots, i_d) \in I^\nu$ and $\underline{j} = (j_1, \dots, j_{d'}) \in I^{\nu'}$, let $\underline{ij} = (i_1, \dots, i_d, j_1, \dots, j_{d'})$. Then, there exists an embedding

$$\iota_{\nu, \nu'} : H(\Gamma; \nu, \nu') \rightarrow H(\Gamma; \nu + \nu') \quad (3.5.1)$$

given by $\iota_{\nu, \nu'}(e(\underline{i}) \otimes e(\underline{j})) = e(\underline{ij})$ and, for appropriate r and s , and for a and b among the symbols y or ϕ ,

$$\iota_{\nu, \nu'}(a_r \otimes b_s) = a_r b_{s+d}.$$

If $M \in \text{Rep}(\Gamma; \nu)$ and $N \in \text{Rep}(\Gamma; \nu')$, let $M \boxtimes N \in \text{Rep}(\Gamma; \nu) \otimes \text{Rep}(\Gamma; \nu')$ denote the *outer* tensor product of M and N . We have

Proposition 3.5.2. [KL1, Proposition 2.16] *We have $\iota_{\nu, \nu'}(1_\nu \otimes 1_{\nu'})H(\Gamma; \nu + \nu')$ is a free graded left $H(\Gamma; \nu, \nu')$ -module.*

Therefore, we may define the exact functors

$$\text{Res}_{\nu, \nu'}^{\nu + \nu'} : H(\Gamma; \nu + \nu') \rightarrow H(\Gamma; \nu, \nu') \quad (3.5.2)$$

by $\text{Res}_{\nu, \nu'}^{\nu + \nu'} M = \iota_{\nu, \nu'}(1_\nu \otimes 1_{\nu'})M$, and

$$\text{Ind}_{\nu, \nu'}^{\nu + \nu'} : H(\Gamma; \nu, \nu') \rightarrow H(\Gamma; \nu + \nu'), \quad (3.5.3)$$

by

$$\text{Ind}_{\nu, \nu'}^{\nu + \nu'} M \boxtimes N = H(\Gamma; \nu + \nu') \otimes_{H(\Gamma; \nu, \nu')} M \boxtimes N.$$

We have

Lemma 3.5.3. [KL1, Lemma 2.20] *Assume that $M \in \text{Rep}(\Gamma; \nu)$, $N \in \text{Rep}(\Gamma; \nu')$,*

$$\text{ch } M = \sum_{\underline{i} \in I^\nu} m_{\underline{i}} [\underline{i}] \quad \text{and} \quad \text{ch } N = \sum_{\underline{j} \in I^{\nu'}} n_{\underline{j}} [\underline{j}].$$

Then,

$$\text{ch } \text{Ind}_{\nu, \nu'}^{\nu + \nu'} M \boxtimes N = \sum_{\underline{i} \in I^\nu, \underline{j} \in I^{\nu'}} m_{\underline{i}} n_{\underline{j}} [\underline{j}] * [\underline{i}]$$

*where $[\underline{j}] * [\underline{i}]$ is the shuffle product given by (2.2.2).*

Remark 3.5.4. *Observe that the order of the segments in the shuffle lemma is reversed. This is a consequence of the definition (2.2.1) and is so that the terms in the character formula coming from $1 \otimes (M \boxtimes N)$ are not shifted in degree. Note that this is slightly different than the shuffle product in [KR2]. The products are related by the formula*

$$x \circ y = y * x$$

for $x, y \in \mathcal{W}$.

Let $\text{Proj}(\Gamma)$ (resp. $\text{Proj}(\Gamma; \nu)$) denote the category of finitely generated, graded, projective (left) $H(\Gamma)$ -modules (resp. $H(\Gamma; \nu)$ -modules). Let $K_0(\Gamma)$ (resp. $K_0(\Gamma; \nu)$) denote Grothendieck group of $\text{Proj}(\Gamma)$ (resp. $\text{Proj}(\Gamma; \nu)$), and set $K_0(\Gamma)_{\mathbb{Q}(q)} = K_0(\Gamma) \otimes_{\mathcal{A}} \mathbb{Q}(q)$. Also, define the analogous functors $\text{Ind}_{\nu, \nu'}^{\nu + \nu'}$ and $\text{Res}_{\nu, \nu'}^{\nu + \nu'}$ to those for $\text{Rep}(\Gamma)$.

Given a left $H(\Gamma; \nu)$ -module M , let M^ψ be the right $H(\Gamma; \nu)$ -module given by $mx = \psi(x)m$ for all $x \in H(\Gamma; \nu)$ and $m \in M$. Define the *Kashiwara-Khovanov-Lauda pairing* (KKL), $(\cdot, \cdot)_{KKL} : K_0(\Gamma; \nu) \times K(\Gamma; \nu) \rightarrow \mathcal{A}$ by

$$([P], [M])_{KKL} = \prod_{i \in I} (1 - q_i^2)^{c_i} \dim_q (P^\psi \otimes M), \quad (3.5.4)$$

if $\nu = \sum_{i \in I} c_i \alpha_i$. This form is evidently related to the *Lusztig-Khovanov-Lauda pairing* (LKL), $(\cdot, \cdot)_{LKL}$, appearing in [KL1, (2.43), (2.44)] by the formula

$$([P], [M])_{KKL} = \prod_{i \in I} (1 - q_i^2)^{c_i} ([P], [M])_{LKL}, \quad (3.5.5)$$

see Remark 2.2.2. Define the map $\omega : K(\Gamma) \rightarrow K_0(\Gamma)_{\mathbb{Q}(q)}$,

$$\omega([M]) = \sum_{[P] \in \mathcal{B}} ([P], [M])_{KKL} [P], \quad (3.5.6)$$

where the sum is over a basis \mathcal{B} of $K_0(\Gamma)$, and $M \in \text{Rep}(\Gamma)$.

Example 3.5.5. *Let $\mathbf{1}_{\alpha_i}$ denote the unique irreducible $H(\Gamma; \alpha_i)$ -module concentrated in degree 0. It is one dimensional with the action of $H(\Gamma; \alpha_i)$ given by $e(j)\mathbf{1}_{\alpha_i} = \delta_{ij}\mathbf{1}_{\alpha_i}$, $y_1\mathbf{1}_{\alpha_i} = \{0\}$. Let \mathcal{P}_{α_i} denote its projective cover. Then,*

$$([\mathcal{P}_{\alpha_i}], [\mathbf{1}_{\alpha_i}])_{KKL} = (1 - q_i^2).$$

In particular, under the identification of $K(\Gamma)$ with $K_0^(\Gamma)$, we have $\omega([\mathbf{1}_{\alpha_i}]) = [\mathcal{P}_{\alpha_i}] - [\mathcal{P}_{\alpha_i}\{2d_i\}]$. That is, $[\mathbf{1}_{\alpha_i}]$ is mapped by ω to its projective resolution*

$$0 \longrightarrow \mathcal{P}_{\alpha_i}\{2d_i\} \longrightarrow \mathcal{P}_{\alpha_i} \longrightarrow \mathbf{1}_{\alpha_i} \longrightarrow 0.$$

More generally, using [KR2, Lemma 3.2], we deduce that if $\mathcal{L} \in \text{Rep}(\Gamma; \nu)$ is a simple module satisfying $\sigma(\text{ch } \mathcal{L}) = \text{ch } \mathcal{L}$, $\mathcal{P}_{\mathcal{L}} \in \text{Proj}(\Gamma; \nu)$ is its projective cover, and $\nu = \sum_i c_i \alpha_i$, then

$$([\mathcal{P}_{\mathcal{L}}], [\mathcal{L}])_{KKL} = \prod_{i \in I} (1 - q_i^2)^{c_i},$$

so $\omega([\mathcal{L}]) = \prod_i (1 - q_i^2)^{c_i} [\mathcal{P}_{\mathcal{L}}]$. On the other hand, if $\mathcal{L}, \mathcal{L}' \in \text{Rep}(\Gamma)$ are two simple modules as above,

$$([\mathcal{P}_{\mathcal{L}}], [\mathcal{L}'])_{LKL} = \delta_{\mathcal{L}, \mathcal{L}'},$$

so identifying $K(\Gamma)$ with the dual lattice to $K_0(\Gamma)$ using the Lusztig-Khovanov-Lauda pairing does not contain any representation theoretic information.

We identify $K(\Gamma)$ with its image under ω . The following lemma shows that this image is the dual lattice

$$K_0^*(\Gamma) = \{X \in K_0(\Gamma)_{\mathbb{Q}(q)} \mid (Y, X)_{LKL} \in \mathcal{A} \text{ for all } Y \in K_0(\Gamma)\},$$

where

$$(\cdot, \cdot)_{LKL} : K_0(\Gamma)_{\mathbb{Q}(q)} \times K_0(\Gamma)_{\mathbb{Q}(q)} \rightarrow \mathbb{Q}(q) \quad (3.5.7)$$

is the Lusztig-Khovanov-Lauda bilinear form, given by $([P], [Q])_{LKL} = \dim_q(P^\psi \otimes_{H(\Gamma; \nu)} Q)$ for $P, Q \in \text{Proj}(\Gamma; \nu)$, cf. [KL1, (2.45), (2.46), (2.47)].

Lemma 3.5.6. *Under the identification above, the simple modules are dual to their projective covers with respect to the Lusztig-Khovanov-Lauda bilinear form. In particular, the map $X \mapsto (\omega(X), ?)_{LKL}$ identifies the dual lattice $K^*(\Gamma; \nu)$ with the dual space $\text{Hom}_{\mathcal{A}}(K_0(\Gamma; \nu), \mathcal{A})$.*

Proof: Let $\nu = \sum_i c_i \alpha_i \in \mathcal{Q}^+$. Assume that $\{\mathcal{L}_a \mid a \in A\}$ is a basis for $K(\Gamma; \nu)$ for some indexing set A , and let \mathcal{B} in (3.5.6) be the basis for $K_0(\Gamma; \nu)$ consisting of the projective covers \mathcal{P}_a of \mathcal{L}_a , $a \in A$. Then, by the definitions $([\mathcal{P}_a], [\mathcal{L}_b])_{KKL} = \delta_{ab} \prod_i (1 - q_i^2)^{c_i}$. Therefore, $\omega([\mathcal{L}_a]) = \prod_i (1 - q_i^2)^{c_i} [\mathcal{P}_a]$. Also, $([\mathcal{P}_b], [\mathcal{P}_a])_{LKL} = \delta_{ba} (1 - q_i^2)^{-c_i}$. Hence,

$$(\omega([\mathcal{L}_b]), [\mathcal{P}_a])_{LKL} = (([\mathcal{P}_b], [\mathcal{L}_b])_{KKL} [\mathcal{P}_b], [\mathcal{P}_a])_{LKL} = ([\mathcal{P}_b], [\mathcal{L}_b])_{KKL} ([\mathcal{P}_b], [\mathcal{P}_a])_{LKL} = \delta_{ba}.$$

■

Identifying $\mathcal{U}_{\mathcal{A}, \nu}^*$ with $\text{Hom}_{\mathcal{A}}(\mathcal{U}_{\mathcal{A}, \nu}, \mathcal{A})$ using Kashiwara's bilinear form, we obtain the following result which is dual to the main results in [KL1, KL2]:

Theorem 3.5.7. [KL1, Theorem 1.1], [KL2, Theorem 8] *In the notation of §2.1-2.2, there is an isomorphism of \mathcal{Q}^+ -graded twisted bialgebras*

$$\gamma^* : K(\Gamma) \rightarrow \mathcal{U}_{\mathcal{A}}^*.$$

Define multiplication $\circ : K(\Gamma; \nu) \otimes K(\Gamma; \nu') \rightarrow K(\Gamma; \nu + \nu')$ by

$$[M] * [N] = [\text{Ind}_{\nu', \nu}^{\nu + \nu'} N \boxtimes M].$$

Define multiplication on $\text{Proj}(\Gamma)$ by $[P] \cdot [Q] = [\text{Ind}_{\nu, \nu'}^{\nu + \nu'} P \boxtimes Q]$. In light of Remark 3.5.4, we have the following slight modification of [KR2, Lemma 3.5]:

Lemma 3.5.8. [KR2, Lemma 3.5] *For $P \in \text{Proj}(\Gamma; \nu + \nu')$, $M \in \text{Rep}(\Gamma; \nu)$ and $N \in \text{Rep}(\Gamma; \nu')$,*

$$([P], [M] * [N])_{KKL} = ([\text{Res}_{\nu', \nu}^{\nu + \nu'} P], [N] \otimes [M])_{KKL}.$$

For $P \in \text{Proj}(\Gamma; \nu)$, $Q \in \text{Proj}(\Gamma; \nu')$, and $M \in \text{Rep}(\Gamma; \nu + \nu')$,

$$([P] \cdot [Q], [M])_{KKL} = ([P] \otimes [Q], [\text{Res}_{\nu, \nu'}^{\nu + \nu'} M])_{KKL}.$$

Remark 3.5.9. *We note that using (3.5.5) does not affect the lemma above, since the renormalization factor on both sides of the equations above is the same.*

Observe that the order of ν and ν' in the first equation of Lemma 3.5.8 have been reversed, but not in the second. This implies that $\gamma^* \circ \text{mult} = \text{mult} \circ (\gamma^* \otimes \gamma^*) \circ \text{flip}$, where mult denotes the appropriate multiplication map, and $\text{flip} : K(\Gamma) \otimes K(\Gamma) \rightarrow K(\Gamma) \otimes K(\Gamma)$ is the map $\text{flip}([M] \otimes [N]) = [N] \otimes [M]$. In particular, we have the following property of γ^* as proved in [KR2].

Theorem 3.5.10. [KR2, Theorem 4.4(5)] *For $[M], [N] \in K(\Gamma)$,*

$$\gamma^*([M] * [N]) = \gamma^*([M])\gamma^*([N]).$$

We also record the following, which was proved in [KR2].

Theorem 3.5.11. [KR2, Theorem 4.4(3)] *The following diagram commutes:*

$$\begin{array}{ccc} K(\Gamma) & \xrightarrow{\gamma^*} & \mathcal{U}_{\mathcal{A}}^* \\ & \searrow \text{ch} & \swarrow \Psi \\ & \mathcal{W}_{\mathcal{A}}^* & \end{array}$$

Proof: It is more convenient to show that $\text{ch} \circ (\gamma^*)^{-1} = \Psi$. To this end, assume that $u \in \mathcal{U}_{\mathcal{A}, \nu}^*$. Then, u may be written as

$$u = \sum n_{i_1, \dots, i_d} e_{i_1} \cdots e_{i_d},$$

where the sum is over all $(i_1, \dots, i_d) \in I^\nu$.

Now, let $\mathbf{1}_{\alpha_i} \in \text{Rep}(\Gamma; \alpha_i)$ be the unique irreducible representation, see Example 3.5.5. It is clear from Theorem 3.5.7 that $\gamma^*([\mathbf{1}_{\alpha_i}]) = e_i$. Therefore,

$$\begin{aligned} \text{ch} \circ (\gamma^*)^{-1}(u) &= \text{ch} \circ (\gamma^*)^{-1} \left(\sum n_{i_1, \dots, i_d} e_{i_1} \cdots e_{i_d} \right) \\ &= \text{ch} \left(\sum n_{i_1, \dots, i_d} [\mathbf{1}_{\alpha_{i_1}}] * \cdots * [\mathbf{1}_{\alpha_{i_d}}] \right) \\ &= \text{ch} \left(\sum n_{i_1, \dots, i_d} [\text{Ind}_{\alpha_{i_1}, \dots, \alpha_{i_d}}^\nu \mathbf{1}_{\alpha_{i_d}} \boxtimes \cdots \boxtimes \mathbf{1}_{\alpha_{i_1}}] \right) \\ &= \sum n_{i_1, \dots, i_d} [i_1] * \cdots * [i_d] \\ &= \Psi(u). \end{aligned}$$

■

Remark 3.5.12. *We point out that Kleshchev and Ram prove several other important properties of the isomorphism γ^* in [KR2, Theorem 4.4]. However, as we do not use these properties, we refer the reader to their paper for the details.*

4. STANDARD REPRESENTATIONS AND THEIR SIMPLE QUOTIENTS

4.1. Cuspidal Representations. Following Kleshchev and Ram, we call a monomial $f \in \mathcal{F}$ a *weight* of $M \in \text{Rep}(\Gamma)$ if $M_{\underline{i}_f} \neq 0$, where $\underline{i}_f \in I^\infty$ is the reading of the word f . That is, $f = [\underline{i}_f]$. Since the set of words in \mathcal{F} is totally ordered, it makes sense to speak of the *lowest weight* of a module.

Fix a (right) Lyndon ordering on Δ^+ . Continuing with the terminology of Kleshchev and Ram, we call an irreducible module *cuspidal* if it has lowest weight $l(\beta) \in \mathcal{GL}$ for some $\beta \in \Delta^+$.

Theorem 4.1.1. *For the (right) Lyndon order on Δ^+ used in Section 5, cuspidal representations exist in all finite types. Moreover, for each $l \in \mathcal{GL}$, $\text{ch } \mathbf{1}_l = b_l^*$.*

Proof: For types $ABCD$, the representations are constructed explicitly in Section 6. We deduce the E_8 case from [KR1, Lemma 3.3, Theorems 3.6, 3.10], since the corresponding Lyndon words are homogeneous. Finally, the G_2 case follows easily from the construction in [KR2] since the characters are identical. ■

4.2. Standard Representations and Unique Irreducible Quotients. We continue to use the ordering from Section 5. Given $g \in \mathcal{G}$, $g = l(\beta_1) \cdots l(\beta_k)$, with $\beta_1 \geq \cdots \geq \beta_k$ define

$$\mathcal{M}(g) = (\text{Ind}_{\beta_1, \dots, \beta_k}^{\beta_1 + \dots + \beta_k} \mathbf{1}_{\beta_1} \boxtimes \cdots \boxtimes \mathbf{1}_{\beta_k}) \{c_g\}.$$

The following is a consequence of Lemma 3.5.3, (2.4.3) and the definition.

Proposition 4.2.1. *For each $g \in \mathcal{G}$,*

$$\text{ch } \mathcal{M}(g) = E_g^*.$$

In particular, $\dim_q \mathcal{M}(g)_{i_g} = \kappa_g$.

The next theorem now follows from the previous proposition using Theorem 3.5.11.

Theorem 4.2.2. *The set*

$$\{[\mathcal{M}(g)] \mid g \in \mathcal{G}\}$$

forms a basis for $K(\Gamma)$.

The following crucial lemma is proved in [KR2].

Lemma 4.2.3. [KR2, Lemma 6.6] *Let $g = l^k$ for some $l = l(\beta) \in \mathcal{GL}_d$, then $\mathcal{M}(g)$ is irreducible.*

The above lemma, together with a Frobenius reciprocity argument yields the main result of [KR2]:

Theorem 4.2.4. [KR2, Theorem 7.2(i)] *Let $g \in \mathcal{G}_d$. Then $\mathcal{M}(g)$ has a unique maximal submodule $\mathcal{R}(g)$ and unique simple quotient $\mathcal{L}(g)$.*

As noted in [KR2], Khovanov and Lauda prove that for every simple module L , there is a unique grading shift such that $\sigma(\text{ch } L\{s\}) = \text{ch } L\{s\}$, [KL1, §3.2]. Therefore, by Theorems 4.1.1 and 2.4.2, and [Le, Proposition 32],

Theorem 4.2.5. [KR2, Theorem 7.2(iii)] *We have $\sigma(\text{ch } \mathcal{L}(g)) = \text{ch } \mathcal{L}(g)$.*

Finally, we have

Theorem 4.2.6. [KR2, Theorem 7.2(iv)] *The set*

$$\{[\mathcal{L}(g)] \mid g \in \mathcal{G}\}$$

forms a basis for $K(\Gamma)$.

4.3. Twisting by the Automorphism τ . Finally, we close by relating the representation theory coming from the (right) Lyndon orderings on Δ^+ to the (left) Lyndon orderings that appear in [KR2]. To this end, given $M \in \text{Rep}(\Gamma)$, let M^τ be the module obtained by twisting by the automorphism τ , cf. Proposition 3.4.1. That is, $M^\tau = M$ as graded vector spaces with $x \cdot m = \tau(x)m$ for all $m \in M^\tau$.

Recall the opposite ordering and related notation developed in §2.5. We have the following:

Theorem 4.3.1. *Let $g \in \mathcal{G}$. Then, $\mathcal{L}(g)^\tau = \mathcal{L}(g^\tau)$.*

Proof: First, it is immediate by character considerations that the cuspidal representations satisfy $\mathbf{1}_l^\tau = \mathbf{1}_{l^\tau}$, see Lemma 2.5.1(4). Therefore, it follows that $\mathcal{M}(g)^\tau = \mathcal{M}(g^\tau)$ for all $g \in \mathcal{G}$. The result now follows since \mathcal{R} is a submodule of $\mathcal{M}(g)$ if, and only if, \mathcal{R}^τ is a submodule of $\mathcal{M}(g^\tau)$. ■

5. IDENTIFICATION OF GOOD LYNDON WORDS AND ASSOCIATED ROOT VECTORS

We now give explicit descriptions of the good Lyndon words and associated root vectors for \mathfrak{g} of classical type and type F_4 . In type E_8 we determine the good Lyndon words. Throughout, we write $b^*[i] := b_{[i]}^*$ for good Lyndon words $l = [i]$.

5.1. Classical Type. We now specialize to the case where \mathfrak{g} is of classical type. Fix a labeling of the simple roots as in Table 5.1.

TABLE 1. Labelling of Simple Roots

Type	Diagram	Positive Roots
A_r		$\alpha_i + \alpha_{i+1} + \cdots + \alpha_j, 0 \leq i \leq j < r.$
B_r		$\alpha_i + \alpha_{i+1} + \cdots + \alpha_j, 0 \leq i \leq j < r,$ $2\alpha_0 + \cdots + 2\alpha_j + \alpha_{j+1} + \cdots + \alpha_k, 0 \leq j < k < r.$
C_r		$\alpha_i + \alpha_{i+1} + \cdots + \alpha_j, 0 \leq i \leq j < r,$ $\alpha_0 + 2\alpha_1 + \cdots + 2\alpha_j + \alpha_{j+1} + \cdots + \alpha_k, 0 \leq j \leq k < r.$
D_r		$\alpha_i + \alpha_{i+1} + \cdots + \alpha_j, 0 \leq i \leq j < r,$ $\alpha_0 + \alpha_2 + \cdots + \alpha_j, 2 \leq j < r,$ $\alpha_0 + \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_j + \alpha_{j+1} + \cdots + \alpha_k, 2 \leq j < k < r.$

We have the following description of good Lyndon words. Calculations can be found in Appendix A.1.

Proposition 5.1.1. *We have*

- (1) *The good Lyndon words for \mathfrak{g} of type A_r are*

$$\{[i, \dots, j] | 0 \leq i \leq j < r\}.$$

- (2) *The good Lyndon words for \mathfrak{g} of type B_r are*

$$\{[i, \dots, j] | 0 \leq i \leq j < r\} \cup \{[j, j-1, \dots, 0, 0, \dots, k-1, k] | 0 \leq j < k < r\}.$$

(3) The good Lyndon words for \mathfrak{g} of type C_r are

$$\{[i, \dots, j] | 0 \leq i \leq j < r\} \cup \{[j, \dots, 1, 0, 1, \dots, k] | 1 \leq j < k \leq r-1\} \cup \{[0, \dots, j, 1, \dots, j] | 1 \leq j < r\}.$$

(4) The good Lyndon words for \mathfrak{g} of type D_r are

$$\{[0, 2, \dots, i] | 2 \leq i < r\} \cup \{[i, \dots, j] | 1 \leq i \leq j \leq r-1\} \cup \{[j, \dots, 1, 0, 2, \dots, k] | 1 \leq j < k < r\}.$$

We now list the root vectors associated to the good Lyndon words. Calculations can be found in Appendix A.2

Proposition 5.1.2. (1) In type A_r ,

$$b^*[i, \dots, j] = [i, \dots, j], \quad 0 \leq i \leq j < r.$$

(2) In type B_r :

$$b^*[i, \dots, j] = [i, \dots, j], \quad 0 \leq i \leq j < r$$

$$b^*[j, \dots, 0, 0, \dots, k] = [2]_0[j, \dots, 0, 0, \dots, k], \quad 0 \leq j < k < r.$$

(3) In type C_r :

$$b^*[i, \dots, j] = [i, \dots, j], \quad 0 \leq i \leq j < r,$$

$$b^*[j, \dots, 1, 0, 1, \dots, k] = [j, \dots, 1, 0, 1, \dots, k], \quad 1 \leq j < k < r,$$

$$b^*[0, \dots, j, 1, \dots, j] = q[0] \cdot ([1, \dots, j] * [1, \dots, j]), \quad 1 \leq j < r.$$

(4) In type D_r :

$$b^*[0] = [0]$$

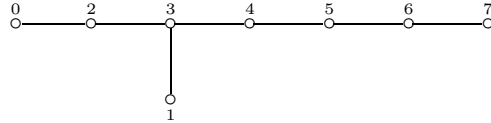
$$b^*[0, 2, \dots, i] = [0, 2, \dots, i], \quad 2 \leq i < r,$$

$$b^*[i, \dots, j] = [i, \dots, j], \quad 1 \leq i \leq j < r,$$

$$b^*_{[1,0,2,\dots,j]} = [1, 0, 2, \dots, j] + [0, 1, 2, \dots, j], \quad 2 \leq j < r,$$

$$b^*[j, \dots, 2, 1, 0, 2, \dots, k] = [j, \dots, 2, 1, 0, 2, \dots, k] + [j, \dots, 2, 0, 1, 2, \dots, k], \quad 2 \leq j < k < r.$$

5.2. Good Lyndon Words in Type E_8 . Fix the following labeling of the nodes of the Dynkin diagram for E_8 .



We list here only the 12 good Lyndon words belonging to the set \mathcal{E} in [KR2, §8.3]:

$$\begin{aligned} & [6023145342302134567], [56023145345342302134567], [45623145342302134567], \\ & [3456023145342302134567], [13456023145342302134567], [23456023145342302134567], \\ & [323131456023145342302134567], [432131456023145342302134567], \\ & [543213456023145342342302134567], [6543213456023145342342302134567], \\ & [53423021345676451342302134567]. \end{aligned}$$

The complete list of the 120 good Lyndon words for E_8 can be found in Appendix A.4.

5.3. Good Lyndon Words and Root Vectors in Type F_4 . We now calculate the Lyndon words and corresponding root vectors for \mathfrak{g} of type F_4 . We choose the following labeling of the Dynkin diagram.



Note that we have the opposite ordering as that in [KR2].

Proposition 5.3.1. *The good Lyndon words for F_4 are given in the following table:*

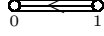
Height	Good Lyndon Words
1	[0], [1], [2], [3]
2	[01], [12], [23]
3	[012], [123], [112]
4	[0123], [1012], [1123]
5	[01012], [21123], [10123]
6	[010123], [210123]
7	[1210123], [2010123]
8	[12010123]
9	[112010123]
10	[2112010123]
11	[21012310123]

Proposition 5.3.2. *The root vectors for F_4 are given as follows:*

$$\begin{aligned}
b^*[i, \dots, j] &= [i, \dots, j] \\
b^*[112] &= [2]_0[112] \\
b^*[1012] &= [1012] + [2]_0[0112] \\
b^*[1123] &= [2]_0[1123] \\
b^*[01012] &= [2]_0[01012] + [2]_0^2[00112] \\
b^*[21123] &= [2]_0[21123] \\
b^*[10123] &= [10123] + [2]_0[01123] \\
b^*[010123] &= [2]_0[010123] + [2]_0^2[001123] \\
b^*[210123] &= [210123] + [2]_0([201123] + [021123]) \\
b^*[1210123] &= [1210123] + [2]_0([1021123] + [1201123]) \\
b^*[2010123] &= [2]_0([2010123] + [0210123]) + [2]_0^2([2001123] + [0201123] + [0021123]) \\
b^*[12010123] &= [2]_0([12010123] + [10210123]) + [01210123] \\
&\quad + [2]_0^2([12001123] + [10201123] + [10021123]) \\
&\quad + [2]_0([01201123] + [01021123]) \\
b^*[112010123] &= [2]_0[1] \cdot b^*[12020123] \\
b^*[2112010123] &= [2] \cdot b^*[112010123] \\
b^*[21012310123] &= q[2] \cdot (b^*[10123] * b^*[10123])
\end{aligned}$$

Proof: For good Lyndon words of height at most 10, the result is obtained by direct calculation. The calculation of the height 11 case is analogous to that for the long roots in type C_r , see Proposition A.2.3 below. ■

5.4. **The type G_2 case.** Fix the following labelling on the Dynkin diagram for G_2 :



Below we list the good Lyndon words, and associated root vectors:

Proposition 5.4.1. *The good Lyndon words for Γ of type G_2 are*

$$[0], [1], [01], [001], [0001], [00101].$$

Proposition 5.4.2. *The root vectors for G_2 are as follows:*

$$\begin{aligned} b^*[0] &= [0], \\ b^*[1] &= [1], \\ b^*[01] &= [01], \\ b^*[001] &= [2]_0[001] \\ b^*[0001] &= [2]_0[3]_0[0001], \\ b^*[00101] &= [2]_0[3]_0[00101] + [2]_0[3]_0[2]_1[00011]. \end{aligned}$$

We note here that the Lyndon words and associated root vectors agree whether we read from right-to-left or from left-to-right, cf. [Le, §5.5.4].

6. CONSTRUCTION OF THE CUSPIDAL REPRESENTATIONS

Fix the (right) Lyndon ordering on \mathcal{G} as in Section 5. Recall that \underline{l} denotes the reading of a good Lyndon word l . That is $l = [\underline{l}]$. Throughout this section, we will need the converse to [KR2, Lemma 6.4]. The proof is very similar to [KR2, Lemma 6.6].

Lemma 6.0.3. *Let $V \in H(\Gamma; \beta)$, and assume that $\text{ch } V = b_l^*$ for some $l = l(\beta) \in \mathcal{GL}$. Then, V is irreducible.*

Proof: By Theorem 2.4.2, all composition factors of V have lowest weight $g \in \mathcal{G}$ for $g > l$. On the other hand, all composition factors have lowest weight belonging to \mathcal{G}_β so, by Corollary 2.4.1, $V = L^{\oplus k}$ for some simple module L . The result now follows because $\{b_g^* \mid g \in \mathcal{G}\}$ is an \mathcal{A} basis of $\mathcal{U}_{\mathcal{A}}^*$. ■

6.1. **Type A_r .** Let $l = [i, \dots, j]$, $0 \leq i \leq j < r$. We have $b_l^* = [i, \dots, j]$. Define $\mathbf{1}_l = \mathbb{F}.v_0$ where v_0 has degree 0. Set $e(\underline{i})\mathbf{1}_l = \delta_{\underline{i}, \underline{l}}\mathbf{1}_l$, $\phi_s v_0 = 0$ and $y_s v_0 = 0$ for all admissible s . This is the trivial representation of $H(\Gamma; \nu)$ and clearly satisfies (3.2.1)-(3.2.11) and $\text{ch } \mathbf{1}_l = b_l^*$.

6.2. Type B_r . The case $l = [i, \dots, j]$, $0 \leq i \leq j < r$ is the trivial representation as in type A_r .

Let $l = [j, \dots, 0, 0, \dots, k]$, $0 \leq j < k < r$. Then, $b_l^* = (q + q^{-1})[j, \dots, 0, 0, \dots, k]$. Set $\mathbf{1}_l = \mathbb{F}v_1 \oplus \mathbb{F}v_{-1}$, where $\deg v_i = i$ for $i = \pm 1$. Define $e(\underline{i})\mathbf{1}_l = \delta_{\underline{i}, \underline{i}_l}\mathbf{1}_l$. Set $y_r v_1 = 0$ for all s , for $s \neq j+1$, set $\phi_s v_1 = 0$, and define $\phi_{j+1} v_1 = v_{-1}$. Set $\phi_r v_{-1} = 0$ for all s , for $s \neq j+1, j+2$, set $\phi_r v_{-1} = 0$, and set $y_{j+1} v_{-1} = -v_1$ and $y_{j+2} v_{-1} = v_1$. We leave it as an easy exercise to the reader to check that this satisfies (3.2.1)-(3.2.11) and $\text{ch } \mathbf{1}_l = b_l^*$.

6.3. Type C_r . For $l \neq [0, \dots, j, 1, \dots, j]$, $\mathbf{1}_l$ is the trivial representations and may be computed as in type A_r .

Assume $l = [0, \dots, j, 1, \dots, j]$. Then $b_l^* = q[0]([1, \dots, j] * [1, \dots, j])$. Let $\beta = \alpha_1 + \dots + \alpha_j$, and consider the $H(\Gamma; \alpha_0, 2\beta)$ module $\mathbf{1}_{\alpha_0} \boxtimes (\text{Ind}_{\beta, \beta}^{2\beta} \mathbf{1}_\beta \boxtimes \mathbf{1}_\beta) \{1\}$. Extend this to a $H(\Gamma; 2\beta + \alpha_0)$ module by insisting that ϕ_1 acts as 0, and $e(\underline{i})$ acts as 0 if $i_1 \neq 0$. It is very easy to check that this is the desired cuspidal representation, $\mathbf{1}_l$, cf. [KR2, §8.6].

6.4. Type D_r . For $l \neq [1, 0, 2, \dots, k]$, $1 \leq j < k < r$, $\mathbf{1}_l$ is the trivial representation and can be computed as in type A_r .

Assume $l = [j, \dots, 1, 0, 2, \dots, k]$. Define $\mathbf{1}_l = \mathbb{F}v_0 \oplus \mathbb{F}w_0$, where v_0 and w_0 have degree 0. Define

$$e(\underline{i})\mathbf{1}_l = \delta_{\underline{i}, \underline{i}_l} \mathbb{F}v_0 + \delta_{\underline{i}, s_j \cdot \underline{i}_l} \mathbb{F}w_0.$$

Define $y_r \mathbf{1}_l = 0$. For $r \neq j$, define $\phi_r \mathbf{1}_l = 0$ and set $\phi_j v_0 = w_0$. It is elementary to check that this is indeed a representation and $\text{ch } \mathbf{1}_l = b_l^*$.

6.5. Type E_8 . We simply note here that in our ordering all Lyndon words for type E_8 are homogeneous in the sense of [KR1] and the corresponding cuspidal representations can be computed using [KR1, Theorems 3.6, 3.10]. The 12 outstanding cases from [KR2] are listed in subsection 5.2 and are evidently homogeneous. An entire list of the good Lyndon words for E_8 can be found in Appendix A.4.

6.6. Type F_4 . We choose the following partial ordering on I : $0 \rightarrow 1 \rightarrow 2 \rightarrow 3$, see (3.1.3).

- (1) $l = [i, \dots, j]$, $0 \leq i \leq j \leq 3$.

Constructed exactly as in the type A case.

- (2) $l = [112], [1123]$, or $[21123]$.

Constructed exactly as in the type B case.

- (3) $l = [1012]$, or $[01012]$.

These are constructed exactly as in the type C case in [KR2]. Indeed, we have

$$b^*[1012] = ([1] * [01])[2], \quad \text{and} \quad b^*[01012] = q([01] * [01])[2].$$

For example, let $\beta = \alpha_0 + \alpha_1$ and define the $H(\Gamma; 2\beta, \alpha_2)$ -module

$$V = (\text{Ind}_{\beta, \beta}^{2\beta} \mathbf{1}_\beta \boxtimes \mathbf{1}_\beta) \boxtimes \mathbf{1}_{\alpha_2} \{1\}.$$

Extend the action to $H(\Gamma; 2\beta + \alpha_2)$ by insisting that ϕ_4 acts as 0 and $e(\underline{i})$ acts as 0 if $i_5 \neq 2$. As in [KR2], the only relation that is not obvious is 3.2.10, which follows since $y_4^2 - y_5$ acts as 0 on the module above. Then, $\mathbf{1}_{[01012]} = V$ is the desired cuspidal representation.

(4) $l = [10123]$ or $[010123]$.

In either case, let $\beta = |l|$. Define the $H(\Gamma; \beta, \alpha_3)$ -module $V = \mathbf{1}_\beta \boxtimes \mathbf{1}_{\alpha_3}$. As above, we may extend this to a $H(\Gamma; \beta + \alpha_3)$ -module by insisting that ϕ_r acts as 0 and $e(\underline{i})$ acts as 0 if $i_{r+1} \neq 3$, where $r = 4$, or 5 as appropriate. To check relation 3.2.10 it is enough to observe that $y_r - y_{r+1}$ acts as 0 on V (actually, each both y_r and y_{r+1} act as 0). Hence, $\mathbf{1}_\beta = V$ is the desired cuspidal representation.

(5) $l = [210123]$.

Let $\beta = |l| - \alpha_2$. Define a graded vector space $V = W \oplus U$, where $W \cong \mathbf{1}_{\alpha_2} \boxtimes \mathbf{1}_\beta$ as a $H(\Gamma; \alpha_2, \beta)$ -module, and $U = U[1] \oplus U[-1]$ is 2-dimensional with basis $\{u_1, u_{-1}\}$. Fix an *weight* basis $\{w_0, w_1, w_{-1}\}$ for W . That is, $e(\underline{i})w_0 = \delta_{\underline{i}, \underline{i}_0} w_0$ and w_0 has degree 0, $w_1 = \phi_2 w_0$, and $w_{-1} = \phi_3 w_1$. It follows from (3.2.11) that $\phi_1 w_{-1} = w_0$.

The following defines an action of $H(\Gamma; \alpha_2 + \beta)$ on V :

- $\phi_1 w_0 = 0$, $\phi_1 w_1 = u_1$, $\phi_1 w_{-1} = u_{-1}$;
- $\phi_2 u_{-1} = 0$;
- $e(\underline{i})$ acts as 0 on W if $i_1 \neq 2$.

Indeed, from (3.2.4) we are forced to define

$$e(\underline{i})u_{\pm 1} = e(\underline{i})\phi_1 w_{\pm 1} = \phi_1 e(s_1 \cdot \underline{i})w_{\pm 1} = \begin{cases} u_{\pm 1} & \text{if } [\underline{i}] = [021123], \\ 0 & \text{otherwise.} \end{cases}$$

Using (3.2.3)-(3.2.9), we must set $y_r u_1 = 0$ for $1 \leq r \leq 6$. For example,

$$y_1 u_1 = y_1 \phi_1 \phi_2 w_0 = \phi_1 y_2 \phi_2 w_0 = \phi_1 \phi_2 y_3 w_0 = 0.$$

Also, we define $y_r u_{-1} = 0$ if $r \neq 3, 4$, and

$$y_3 u_{-1} = y_3 \phi_1 \phi_3 \phi_2 w_0 = \phi_1 (\phi_3 y_4 - 1) \phi_2 w_0 = -u_1.$$

Similarly, $y_4 u_{-1} = u_1$.

Using (3.2.7), we define $\phi_3 u_1 = \phi_1 w_{-1} = u_{-1}$, and $\phi_4 u_{\pm 1} = \phi_5 u_{\pm 1} = 0$. The relation (3.2.10) forces $\phi_1 u_1 = w_1$, $\phi_1 u_{-1} = w_{-1}$, and $\phi_3 u_{-1} = 0$. Using (3.2.11) we define

$$\phi_2 u_1 = \phi_2 \phi_1 \phi_2 w_0 = \phi_1 \phi_2 \phi_1 w_0 = 0.$$

We need to show that the actions of ϕ_1 and ϕ_2 are consistent with relations (3.2.1)-(3.2.11). As explained above, relations (3.2.1)-(3.2.7) follow by definition, as do the relations (3.2.8)-(3.2.10) for the action of ϕ_1 .

We will postpone checking (3.2.11) until we have checked the action of ϕ_2 on U . To check relations (3.2.8) and (3.2.9) we need only consider the nontrivial cases $r = 3, 4$. Indeed, we compute

$$\begin{aligned} y_4 \phi_2 u_{-1} &= y_4 \phi_2 (\phi_1 \phi_3 \phi_2) w_0 \\ &= \phi_2 \phi_1 (\phi_3 y_3 + 1) \phi_2 w_0 && \text{by (3.2.8) in } W \\ &= \phi_2 \phi_1 \phi_3 \phi_2 y_3 w_0 + \phi_2 \phi_1 \phi_2 w_0 \\ &= \phi_1 \phi_2 \phi_1 w_0 && \text{by (3.2.11) in } W \\ &= 0, \end{aligned}$$

since $\phi_1 w_0 = y_2 w_0 = 0$. A similar computation with $r = 3$ gives (3.2.9). To check relation (3.2.10) we need only observe $y_1 u_{-1} = y_2 u_{-1} = 0$. Finally, the last nontrivial relation is

$$\begin{aligned}\phi_1 \phi_2 u_{-1} &= \phi_1 \phi_2 \phi_1 \phi_3 \phi_2 w_0 = \phi_2 \phi_1 \phi_2 \phi_3 \phi_2 w_0 \\ &= \phi_2 \phi_1 (\phi_3 \phi_2 \phi_3 - 1) w_0 = 0.\end{aligned}$$

One has $\text{ch } V = b^*[210123]$. Hence, $\mathbf{1}_l = V$ is the desired representation.

(6) $l = [1210123]$.

Let $\beta = |l| - \alpha_1$ and define the $H(\Gamma; \alpha_1, \beta)$ -module $V = \mathbf{1}_{\alpha_1} \boxtimes \mathbf{1}_\beta$. Extend this to an action of $H(\Gamma; \beta + \alpha_1)$ by insisting that ϕ_1 acts as 0 and $e(\underline{i})$ acts as 0 if $i_1 \neq 1$. Again, the only thing nontrivial to check is (3.2.10) which follows since $y_1^2 - y_2$ acts as 0 on V (actually, both y_1 and y_2 act as 0). Then, $\mathbf{1}_l = V$ is the desired representation.

(7) $l = [2010123]$.

Let $\beta = |l| - \alpha_2$. Define the graded vector space $V = W \oplus U$, where $W \cong \mathbf{1}_{\alpha_2} \boxtimes \mathbf{1}_\beta$ as a $H(\Gamma; \alpha_2, \beta)$ -module and $U = U[2] \oplus U[1] \oplus U[0] \oplus U[-1] \oplus U[-2]$ is 10-dimensional with basis $\{u_2^1, u_2^2, u_1, u_0^1, u_0^2, u_{-0}^1, u_{-0}^2, u_{-1}, u_{-2}^1, u_{-2}^2\}$. Fix a weight basis $\{w_2, w_1, w_0, w_{-0}, w_{-1}, w_{-2}\}$ for W . That is, $e(\underline{i})w_1 = \delta_{\underline{i}, \underline{i}_1} w_1$, $\deg w_1 = 1$, $w_2 = \phi_3 w_1$, $w_0 = \phi_2 w_2$, $w_{-0} = \phi_4 w_2$, $w_{-2} = \phi_4 w_0 = \phi_2 w_{-0}$, and $w_{-1} = \phi_3 w_{-2}$.

The following defines an action of $H(\Gamma, \beta + \alpha_2)$ on V :

- $\phi_1 w_{\pm 1} = u_{\pm 1}$;
- For $i \in \{2, 0, -0, -2\}$, $\phi_1 w_i = u_i^1$;
- $\phi_2 u_1 = \phi_2 u_{-1} = 0$;
- For $i \in \{2, 0, -0, -2\}$, $\phi_2 u_i^1 = u_i^2$;
- $\phi_3 u_{-0}^2 = \phi_3 u_{-2}^2 = 0$
- $e(\underline{i})$ acts as 0 on W if $i_1 \neq 2$.

The remaining relations are now forced. By (3.2.4) we have

- $e(\underline{i})u_{\pm 1} = \begin{cases} u_{\pm 1} & \text{if } \underline{i} = [0210123], \\ 0 & \text{otherwise;} \end{cases}$
- For $i \in \{2, 0, -0, -2\}$, $e(\underline{i})u_i^1 = \begin{cases} u_i^1 & \text{if } \underline{i} = [0201123], \\ 0 & \text{otherwise;} \end{cases}$
- For $i \in \{2, 0, -0, -2\}$, $e(\underline{i})u_i^2 = \begin{cases} u_i^2 & \text{if } \underline{i} = [0021123], \\ 0 & \text{otherwise.} \end{cases}$

We now use (3.2.6) and (3.2.8)-(3.2.9) to define the action of y_1, \dots, y_7 on U . Since y_4, \dots, y_7 commute with ϕ_1 and ϕ_2 , their actions are determined by W . As an example, we compute the action of y_1 on U below. The action of y_2 and y_3 can be worked out similarly.

$$\begin{aligned}y_1 u_2^1 &= \phi_1 y_2 w_2, & y_1 u_2^2 &= \phi_2 (\phi_1 y_2 - 1) w_2, & y_1 u_1 &= \phi_1 y_2 w_1, \\ y_1 u_0^1 &= \phi_1 y_2 w_0, & y_1 u_0^2 &= \phi_2 (\phi_1 y_2 - 1) w_0, & y_1 u_{-0}^1 &= \phi_1 y_2 w_{-0}, \\ y_1 u_{-0}^2 &= \phi_2 (\phi_1 y_2 - 1) w_{-0}, & y_1 u_{-1} &= \phi_1 y_2 w_{-1}, & y_1 u_{-2}^1 &= \phi_1 y_2 w_{-2}, \\ y_1 u_{-2}^2 &= \phi_2 (\phi_1 y_2 - 1) w_{-2}.\end{aligned}$$

Next, to define the action of ϕ_1, \dots, ϕ_6 on U , we note that since ϕ_3, \dots, ϕ_6 commute with ϕ_1 , their actions on $u_2^1, u_1, u_0^1, u_{-0}^1, u_{-1}^1$ and u_{-2}^1 are determined by W . Additionally, since

ϕ_4, ϕ_5, ϕ_6 commute with ϕ_1 and ϕ_2 , their action on u_2^2 , u_0^2 , u_{-0}^2 , and u_{-2}^2 are determined by W . The remaining calculations are given below and can be worked out by rewriting the u 's in the form $\phi_\sigma w_1$.

$$\begin{aligned}\phi_1 u_1 &= w_1, & \phi_1 u_{-1} &= w_{-1}, \\ \phi_1 u_2^1 &= w_2, & \phi_1 u_0^1 &= w_0, & \phi_1 u_{-0}^1 &= w_{-0}, & \phi_1 u_{-2}^1 &= w_{-2} \\ \phi_1 u_2^2 &= u_1^2, & \phi_1 u_0^2 &= 0, & \phi_1 u_{-0}^2 &= u_0^2, & \phi_1 u_{-2}^2 &= 0 \\ \phi_2 u_2^2 &= u_2^1, & \phi_2 u_0^2 &= u_0^1, & \phi_2 u_{-0}^2 &= u_{-0}^1, & \phi_2 u_{-2}^2 &= u_{-2}^1 \\ \phi_3 u_2^2 &= 0, & \phi_3 u_0^2 &= 0,\end{aligned}$$

We now have to check that the actions of ϕ_2 on $u_{\pm 1}$ and ϕ_3 on u_{-0}^2, u_{-2}^2 are consistent with the relations. Indeed, in the case $\phi_2 u_1 = 0$, the only nontrivial relations to check are (3.2.10) and (3.2.11). We have for (3.2.10),

$$\begin{aligned}\phi_2^2 u_1 &= Q_{21}(y_2, y_3) u_1 \\ &= (y_3^2 - y_2) \phi_1 w_1 \\ &= \phi_1 (y_3^2 - y_1) w_1 \\ &= 0.\end{aligned}$$

For the braid relations, we have

$$\phi_1 \phi_2 u_1 = \phi_1 \phi_2 \phi_1 w_1 = \phi_2 \phi_1 \phi_2 w_1 = 0,$$

and

$$\phi_2 \phi_3 \phi_2 u_1 = \phi_3 \phi_2 \phi_3 \phi_1 w_1 = \phi_3 \phi_2 \phi_1 \phi_3 w_1 = \phi_3 u_2^2 = 0.$$

We now check that $\phi_2 u_{-1} = 0$ is consistent with the relations. Indeed, one verifies that

$$\phi_2^2 u_{-1} = Q_{21}(y_2, y_3) u_{-1} = (y_3^2 - y_2) u_{-1} = 0$$

For the braid relations, we have

$$\phi_1 \phi_2 u_{-1} = \phi_1 \phi_2 \phi_1 w_{-1} = \phi_2 \phi_1 \phi_2 w_{-1} = 0,$$

and

$$\begin{aligned}\phi_2 \phi_3 \phi_2 u_{-1} &= \phi_3 \phi_2 \phi_3 u_{-1} \\ &= (\phi_3 \phi_2 \phi_1) \phi_3^2 w_{-2} \\ &= (\phi_3 \phi_2 \phi_1) Q_{01}(y_3, y_4) w_{-2} \\ &= (\phi_3 \phi_2 \phi_1) (y_3 - y_4) \phi_4 \phi_2 \phi_3 w_1 \\ &= \phi_3 \phi_2 \phi_1 (\phi_4 \phi_3 + \phi_2 \phi_3) w_1 \\ &= \phi_3 (u_{-0}^2 + u_0^2) = 0.\end{aligned}$$

We now check the action of ϕ_3 . Indeed, for (3.2.10), a calculation gives

$$\phi_3^2 u_{-0}^2 = Q_{21}(y_3, y_4) u_{-0}^2 = (y_4^2 - y_3) u_{-0}^2 = 0.$$

Similarly, $\phi_3^2 u_{-2}^2 = 0$. For (3.2.11), we need only calculate

$$\begin{aligned}
\phi_2 \phi_3 u_{-0}^2 &= \phi_2 \phi_3 \phi_2 u_{-0}^1 \\
&= \phi_3 \phi_2 \phi_3 (\phi_1 \phi_4 \phi_3) w_1 \\
&= \phi_3 \phi_2 \phi_1 (\phi_4 \phi_3 \phi_4 - 1) w_1 \\
&= -\phi_3 \phi_2 \phi_1 w_1 \\
&= -\phi_3 \phi_2 u_1 = 0.
\end{aligned}$$

Similarly, we have $\phi_2 \phi_3 u_{-2}^1 = 0$.

We have $\text{ch } V = b^*[2010123]$. Hence, $\mathbf{1}_l = V$ is the desired representation.

(8) $l = [12010123]$.

Let $\beta = |l| - \alpha_1$. Define the graded vector space $V = (W \oplus U) \oplus Z$, where $W \oplus U \cong \mathbf{1}_{\alpha_1} \boxtimes \mathbf{1}_\beta$ as a $H(\Gamma; \alpha_1, \beta)$ -module and has a basis as described in the previous case and $Z = Z[1] \oplus Z[0] \oplus Z[-1]$ is 5-dimensional with basis $\{z_1^1, z_1^2, z_0, z_{-1}^1, z_{-1}^2\}$.

The following defines an action of $H(\Gamma; \beta + \alpha_1)$ on V :

- $\phi_1 u_0^1 = z_1^1$;
- $\phi_1 u_0^2 = z_1^2$;
- $\phi_1 u_{-1} = z_0$;
- $\phi_1 u_{-2}^1 = z_{-1}^1$;
- $\phi_1 u_{-2}^2 = z_{-1}^2$;
- ϕ_1 acts as 0 on the remaining basis vectors of $W \oplus U$;
- $e(\underline{i})$ act as 0 on $W \oplus U$ if $i_1 \neq 1$.

We now determine the remaining actions of $H(\Gamma; \beta + \alpha_1)$ on Z . Indeed, note that y_3, \dots, y_8 commute with ϕ_1 , so their actions are determined by $W \oplus U$. To calculate the action of y_1 and y_2 , note that as operators on Z , $y_1 \phi_1 = \phi_1 y_2$ and $y_2 \phi_1 = \phi_1 y_1$ so their action is determined by U . In particular, y_2 acts as 0 on Z since y_1 acts as 0 on U . Additionally, a calculation gives

$$\begin{aligned}
y_1 z_1^1 &= -\phi_1 u_2^1 = 0, & y_1 z_1^2 &= \phi_1 u_2^2 = 0, & y_1 z_0 &= \phi_1 u_1 = 0, \\
y_1 z_{-1}^1 &= -\phi_1 u_{-0}^1 = 0, & y_1 z_{-1}^2 &= -\phi_1 u_{-0}^2 = 0.
\end{aligned}$$

Next observe that the action of ϕ_3, \dots, ϕ_8 on Z are determined by $W \oplus U$. We calculate

$$\phi_1 z_1^1 = -u_2^1, \quad \phi_1 z_1^2 = -u_2^2, \quad \phi_1 z_0 = 0, \quad \phi_1 z_{-1}^1 = -u_{-0}^1, \quad \phi_1 z_{-1}^2 = -u_{-0}^2 \quad (6.6.1)$$

and ϕ_2 acts as 0 on Z .

It remains to check the consistency of this action with the relations. The only relations which are not obvious are (3.2.10) and (3.2.11) for ϕ_1 .

To check (3.2.10) on W it is enough to check that $\phi_1^2 w_1 = Q_{12}(y_1, y_2) w_1 = 0$ which is obvious. Many of the quadratic relations for the action of ϕ_1 on U are contained in (6.6.1) above. The remaining calculation are outlined below.

$$\begin{aligned}
\phi_1^2 u_2^1 &= Q_{10}(y_1, y_2) u_2^1 = 0, & \phi_1^2 u_2^2 &= \phi_3 \phi_1^2 u_2^1 = 0, & \phi_1^2 u_1 &= Q_{10}(y_1, y_2) u_1 = 0, \\
\phi_1^2 u_{-0}^1 &= Q_{10}(y_1, y_2) u_{-0}^1 = 0 & \phi_1^2 u_{-0}^2 &= \phi_3 \phi_1^2 u_{-0}^1 = 0.
\end{aligned}$$

Relation (3.2.10) for the action of ϕ_1 on Z is now obvious.

To check (3.2.11) we need to show that $\phi_1\phi_2z = 0$ for all $z \in Z$. This calculation, however, is trivial. For example,

$$\phi_1\phi_2z_1^1 = \phi_1\phi_2\phi_1u_0^1 = \phi_2\phi_1\phi_2u_0^1 = \phi_2\phi_1\phi_2^2w_0 = \phi_2\phi_1w_0 = 0.$$

We have $\text{ch } V = b^*[12010123]$. Hence, $\mathbf{1}_l = V$ is the desired cuspidal representation.

(9) $l = [112010123]$.

Let $\beta = |l| - \alpha_1$. Define the graded vector space $V = W\{1\} \oplus W\{-1\}$, where $W \cong \mathbf{1}_{\alpha_1} \boxtimes \mathbf{1}_\beta$ as a $H(\Gamma; \alpha_1, \beta)$ -module. For each $w \in W$ write $w\{\pm 1\} \in W\{\pm 1\}$ for the corresponding vector.

The following defines an action of $H(\Gamma; \beta + \alpha_1)$ on V :

- $\phi_1w\{1\} = w\{-1\}$ for $w = e(1, 1, \dots)w$ and $\phi_1w\{1\} = 0$ if $w = e(1, 0, \dots)w$;
- ϕ_1 acts as 0 on $W\{-1\}$;
- $e(\underline{i})$ acts as 0 on V unless $i_1 = 1$.

Once again, it is straightforward to see that this is an $H(\Gamma; \beta + \alpha_1)$ -module. Indeed, the only relation to check is (3.2.10). For $v \in V$, note that either $v = e(11 \dots)v$ or $v = e(10 \dots)v$. Hence

$$\phi_1^2v = \begin{cases} 0 & \text{if } v = e(11 \dots)v; \\ (y_2 - y_1)v & \text{if } v = e(10 \dots)v. \end{cases}$$

The result now follows since both y_1 and y_2 act as 0 on V . We have $\text{ch } V = b^*[112010123]$, so $\mathbf{1}_l = V$ is the desired representation.

(10) $l = [2112010123]$.

Let $\beta = |l| - \alpha_2$ and define $V = \mathbf{1}_{\alpha_2} \boxtimes \mathbf{1}_\beta$ as a $H(\Gamma; \alpha_2, \beta)$ -module. Extend this to an $H(\Gamma; \beta + \alpha_2)$ -module by insisting that ϕ_1 acts as 0 and $e(\underline{i})$ acts as 0 unless $i_1 = 2$. The only relation that is nontrivial to check is (3.2.10), which follows since $y_2^2 - y_1$ acts as 0 on V . Hence $\mathbf{1}_l = V$ is the desired representation.

(11) $l = [21012310123]$.

Let $\beta = \alpha_0 + 2\alpha_1 + \alpha_2 + \alpha_3$. Consider the $H(\Gamma; \alpha_2, 2\beta)$ module

$$V = \mathbf{1}_{\alpha_2} \boxtimes (\text{Ind}_{\beta, \beta}^{2\beta} \mathbf{1}_\beta \boxtimes \mathbf{1}_\beta) \{1\}.$$

Extend this to a $H(\Gamma; 2\beta + \alpha_2)$ module by insisting that ϕ_1 acts as 0 and $e(\underline{i})$ acts as 0 if $i_1 \neq 2$. As in the case of the long roots of type C , clearly the relations for $H(\Gamma; \alpha_2, 2\beta)$ are satisfied. The only new relation which is not obvious is (3.2.10), which follows since $y_2^2 - y_1$ acts as 0 on V . Hence we have constructed a module $\mathbf{1}_l = V$ with character $b^*[21012310123]$.

APPENDIX A. CALCULATIONS

A.1. Good Lyndon Words.

Proposition A.1.1. *We have*

- (1) *The Good Lyndon words for \mathfrak{g} of type A_r are*

$$\{[i, \dots, j] \mid 0 \leq i \leq j < r\}.$$

(2) The good Lyndon words for \mathfrak{g} of type B_r are

$$\{[i, \dots, j] | 0 \leq i \leq j < r\} \cup \{[j, j-1, \dots, 0, 0, \dots, k-1, k] | 0 \leq j < k < r\}.$$

(3) The good Lyndon words for \mathfrak{g} of type C_r are

$$\{[i, \dots, j] | 0 \leq i \leq j < r\} \cup \{[j, \dots, 1, 0, 1, \dots, k] | 1 \leq j < k \leq r-1\} \cup \{[0, \dots, j, 1, \dots, j] | 1 \leq j < r\}.$$

(4) The good Lyndon words for \mathfrak{g} of type D_r are

$$\{[0, 2, \dots, i] | 2 \leq i < r\} \cup \{[i, \dots, j] | 1 \leq i \leq j \leq r-1\} \cup \{[j, \dots, 1, 0, 2, \dots, k] | 1 \leq j < k < r\}.$$

Proof: Proceed by induction on the $\text{ht}(\beta)$. In all types, $\alpha_i \in \Pi$ implies $l(\alpha_i) = [i]$.

(1) For $\beta = \alpha_i + \dots + \alpha_j$, we have

$$C(\beta) = \{(\alpha_i + \dots + \alpha_k, \alpha_{k+1} + \dots + \alpha_j) : j > k \geq i\}.$$

By induction, we assume

$$l(\alpha_{k+1} + \dots + \alpha_j) = [k+1, \dots, j] \quad \text{and} \quad l(\alpha_i + \dots + \alpha_k) = [i, \dots, k].$$

Thus, $l(\beta) = \min \{[i, \dots, k, k+1, \dots, j] : j > k \geq i\} = [i, \dots, j]$ completing our induction.

(2) For $\beta = \alpha_i + \dots + \alpha_j$, we repeat the argument for type A to obtain $l(\beta) = [i, \dots, j]$.

We now calculate $l(\beta)$ for $\beta = 2\alpha_0 + \dots + 2\alpha_j + \alpha_{j+1} + \dots + \alpha_k$. We have

$$\begin{aligned} C(\beta) = & \{(2\alpha_0 + \dots + 2\alpha_j + \alpha_{j+1} + \dots + \alpha_i, \alpha_{i+1} + \dots + \alpha_k) | k > i > j\} \\ & \cup \{(2\alpha_0 + \dots + 2\alpha_i + \alpha_{i+1} + \dots + \alpha_j, \alpha_{i+1} + \dots + \alpha_k) | k > j > i\} \\ & \cup \{(\alpha_{i+1} + \dots + \alpha_j, 2\alpha_0 + \dots + 2\alpha_i + \alpha_{i+1} + \dots + \alpha_k) | k > j > i\} \\ & \cup \{(\alpha_0 + \dots + \alpha_j, \alpha_0 + \dots + \alpha_k)\}. \end{aligned}$$

Recall that $l(\alpha_{i+1} + \dots + \alpha_k) = [i+1, \dots, k]$, $l(\alpha_{i+1} + \dots + \alpha_j) = [i+1, \dots, j]$, $l(\alpha_0 + \dots + \alpha_j) = [0, \dots, j]$ and $l(\alpha_0 + \dots + \alpha_k) = [0, \dots, k]$.

Our base case is $2\alpha_0 + \alpha_1 = \beta$. Here, the first three sets which constitute $C(\beta)$ are empty and $l(\beta) = l(\alpha_0)l(\alpha_0 + \alpha_1) = [0, 0, 1]$.

Assume by induction on the height of β that

$$\begin{aligned} l(2\alpha_0 + \dots + 2\alpha_j + \alpha_{j+1} + \dots + \alpha_i) &= [j, \dots, 0, 0, \dots, i], \quad j < i < k, \\ l(2\alpha_0 + \dots + 2\alpha_i + \alpha_{i+1} + \dots + \alpha_j) &= [i, \dots, 0, 0, \dots, j], \quad i < j, \\ l(2\alpha_0 + \dots + 2\alpha_i + \alpha_{i+1} + \dots + \alpha_k) &= [i, \dots, 0, 0, \dots, k], \quad i < j. \end{aligned}$$

Then,

$$\begin{aligned} l(\beta) &= \min \{[j, \dots, 0, 0, \dots, i, i+1, \dots, k] | i > j\} \\ &\cup \{[i, \dots, 0, 0, \dots, j, i+1, \dots, k] | j > i\} \\ &\cup \{[i+1, \dots, j, i, \dots, 0, 0, \dots, k] | j > i\} \\ &\cup \{[0, \dots, j, 0, \dots, k]\} \\ &= [j, \dots, 0, 0, \dots, k] \end{aligned}$$

completing our induction.

- (3) For $\beta = \alpha_i + \cdots + \alpha_j$, we repeat the argument for type A to obtain $l(\beta) = [i, \dots, j]$.

The next cases are somewhat more subtle. Observe for the base case that

$$C(\alpha_0 + 2\alpha_1) = \{(\alpha_0 + \alpha_1, \alpha_1)\}$$

so that $l(\alpha_0 + 2\alpha_1) = [0, 1, 1]$. Also,

$$C(\alpha_0 + 2\alpha_1 + \alpha_2) = \{(\alpha_0 + 2\alpha_1, \alpha_2), (\alpha_0 + \alpha_1, \alpha_1 + \alpha_2), (\alpha_1, \alpha_0 + \alpha_1 + \alpha_2)\}.$$

Evidently, this gives $l(\alpha_0 + 2\alpha_1 + \alpha_2) = [1, 0, 1, 2]$.

Assume that $\beta = \alpha_0 + 2\alpha_1 + \cdots + 2\alpha_j$, and we have shown that for $i < k \leq j$,

$$l(\alpha_0 + 2\alpha_1 + \cdots + 2\alpha_i) = [0, \dots, i, 1, \dots, i]$$

and

$$l(\alpha_0 + 2\alpha_1 + \cdots + 2\alpha_i + \alpha_{i+1} + \cdots + \alpha_k) = [i, \dots, 1, 0, 1, \dots, k].$$

Observe

$$\begin{aligned} C(\beta) = & \{(\alpha_0 + 2\alpha_1 + \cdots + 2\alpha_i + \alpha_{i+1} + \cdots + \alpha_j, \alpha_{i+1} + \cdots + \alpha_j) \mid 1 \leq i < j\} \\ & \cup \{(\alpha_0 + \cdots + \alpha_j, \alpha_1, \dots, \alpha_j)\}. \end{aligned}$$

Thus,

$$\begin{aligned} l(\beta) = & \min\{[i, \dots, 1, 0, 1, \dots, j, i+1, \dots, j] \mid 1 \leq i < j\} \\ & \cup \{[0, \dots, j, 1, \dots, j]\} \\ = & [0, \dots, j, 1, \dots, j]. \end{aligned}$$

Finally, assume $k > j$ and $\beta = \alpha_0 + 2\alpha_1 + \cdots + 2\alpha_j + \alpha_{j+1} + \cdots + \alpha_k$. Assume further that for all $j < i < k$

$$l(\alpha_0 + 2\alpha_1 + \cdots + 2\alpha_j + \alpha_{j+1} + \cdots + \alpha_i) = [j, \dots, 1, 0, 1, \dots, i],$$

and assume that for either $i < j$ and $m \leq k$, or $i = j$ and $m < k$ that

$$l(\alpha_0 + 2\alpha_1 + \cdots + 2\alpha_i + \alpha_{i+1} + \cdots + \alpha_m) = [i, \dots, 1, 0, 1, \dots, m].$$

We have

$$\begin{aligned} C(\beta) = & \{(\alpha_0 + 2\alpha_1 + \cdots + 2\alpha_j, \alpha_{j+1} + \cdots + \alpha_k)\} \\ & \cup \{(\alpha_0 + 2\alpha_1 + \cdots + 2\alpha_j + \alpha_{j+1} + \cdots + \alpha_i, \alpha_{i+1} + \cdots + \alpha_k) \mid k > i > j\} \\ & \cup \{(\alpha_0 + 2\alpha_1 + \cdots + 2\alpha_i + \alpha_{i+1} + \cdots + \alpha_j, \alpha_{i+1} + \cdots + \alpha_k) : k > j > i\} \\ & \cup \{(\alpha_{i+1} + \cdots + \alpha_j, \alpha_0 + 2\alpha_1 + \cdots + 2\alpha_i + \alpha_{i+1} + \cdots + \alpha_k) : k > j > i\} \end{aligned}$$

Therefore,

$$\begin{aligned}
l(\beta) &= \min \{[0, 1, \dots, j, 1, \dots, k]\} \\
&\cup \{[j, \dots, 1, 0, 1, \dots, k]\} \\
&\cup \{[i, \dots, 1, 0, 1, \dots, j, i+1, \dots, k] \mid j > i > k\} \\
&\cup \{[i+1, \dots, j, i, \dots, 1, 0, 1, \dots, k] \mid j > i > k\} \\
&= [j, \dots, 1, 0, 1, \dots, k].
\end{aligned}$$

(4) Arguing as in the type A case gives $l(\alpha_i + \dots + \alpha_j) = [i, \dots, j]$ for $1 \leq i \leq j$.

Observe that the remaining roots may be written as $\beta = \alpha_0 + \dots + \alpha_j + \alpha_2 + \dots + \alpha_k$ for $0 \leq j < k$ and $k \geq 2$. For the base case we have that $l(\alpha_0 + \alpha_2) = [0, 2]$.

Now, let $\beta = \alpha_0 + \dots + \alpha_j + \alpha_2 + \dots + \alpha_k$, $0 \leq j < k$, $k \geq 2$ (assume $k > 2$ if $j = 0$).

We may assume by induction that if either $i < j$ and $m \leq k$, or $i = j$ and $m < k$ that

$$l(\alpha_0 + \dots + \alpha_i + \alpha_2 + \dots + \alpha_m) = [i, \dots, 1, 0, 2, \dots, m].$$

We have

$$\begin{aligned}
C(\beta) &= \{(\alpha_0 + \alpha_1 + \dots + \alpha_j + \alpha_2 + \dots + \alpha_i, \alpha_{i+1} + \dots + \alpha_k) : k > i > j, i \geq 2\} \\
&\cup \{(\alpha_0 + \alpha_1 + \dots + \alpha_i + \alpha_2 + \dots + \alpha_j, \alpha_{i+1} + \dots + \alpha_k) : k > j > i \geq 0, j \geq 2\} \\
&\cup \{(\alpha_{i+1} + \dots + \alpha_j, \alpha_0 + \alpha_1 + \dots + \alpha_i + \alpha_2 + \dots + \alpha_k) : k > j > i \geq 0\}
\end{aligned}$$

Thus,

$$\begin{aligned}
l(\beta) &= \min \{[j, \dots, 1, 0, 2, \dots, k]\} \\
&\cup \{[i, \dots, 1, 0, 2, \dots, j, i+1, \dots, k] \mid k > j > i \geq 0, j \geq 2\} \\
&\cup \{[i+1, \dots, j, i, \dots, 1, 0, 2, \dots, k] \mid k > j > i \geq 0\} \\
&= [j, \dots, 1, 0, 2, \dots, k]
\end{aligned}$$

■

A.2. Root Vectors.

Proposition A.2.1. *In type A_r ,*

$$b^*[i, \dots, j] = [i, \dots, j], \quad 0 \leq i \leq j < r.$$

Proof: Proceed by induction on $j - i$, the case $j - i = 0$ being trivial. Assume that $i < j$ and $r_{[i+1, \dots, j]} = (q - q^{-1})^{j-i-1} [i + 1, \dots, j]$. Using equation (2.2.3) we deduce that

$$\begin{aligned}
 r_{[i, \dots, j]} &= \Xi(\langle [i, \dots, j] \rangle) = \Xi([i], \langle [i + 1, \dots, j] \rangle)_q \\
 &= \Xi([i]) * \Xi(\langle [i + 1, \dots, j] \rangle) - q^{-1} \Xi(\langle [i + 1, \dots, j] \rangle) * \Xi([i]) \\
 &= [i] * r_{[i+1, \dots, j]} - q^{-1} r_{[i+1, \dots, j]} * [i] \\
 &= (q - q^{-1})^{j-i-1} [i] * [i + 1, \dots, j] - [i] \overline{*} [i + 1, \dots, j] \\
 &= (q - q^{-1})^{j-i-1} ([i] * [i + 1] - [i] \overline{*} [i + 1]) [i + 2, \dots, j] \\
 &= (q - q^{-1})^{j-i-1} (q[i, i + 1] - q^{-1}[i, i + 1]) [i + 2, \dots, j] \\
 &= (q - q^{-1})^{j-i} [i, \dots, j].
 \end{aligned}$$

Finally, using 2.4.9 we deduce that $b^*[i, \dots, j] = E^*[i, \dots, j] = [i, \dots, j]$. ■

Proposition A.2.2. *In type B_r :*

$$\begin{aligned}
 b^*[i, \dots, j] &= [i, \dots, j], \quad 0 \leq i \leq j < r \\
 b^*[j, \dots, 0, 0, \dots, k] &= [2]_0[j, \dots, 0, 0, \dots, k], \quad 0 \leq j < k < r.
 \end{aligned}$$

Proof: The first formula follows easily by induction on $j - i$ as in the type A case. We prove the second formula by induction on j and k with $j < k$, using (2.2.1), (2.3.2), and (2.3.3) for the computations.

Observe that for $k \geq 1$, $r_{[0, 1, \dots, k]} = (q^2 - q^{-2})^k [0, 1, \dots, k]$, which can be proved easily by downward induction on j , $0 \leq j < k$, using (2.2.1) and

$$r_{[j, \dots, k]} = \Xi(\langle [j, \dots, k] \rangle) = \Xi([j], \langle [j + 1, \dots, k] \rangle)_q = [j] * r_{[j+1, \dots, k]} - q^{-2} r_{[j+1, \dots, k]} * [j].$$

By (2.2.1), we have

$$\begin{aligned}
 [0] * [0, 1] - [0, 1] * [0] &= [0, 1, 0] + q^2([0] * [0]) \cdot [1] - ([0] * [0]) \cdot [1] - [0, 1, 0] \\
 &= (q^2 - 1)([0, 0] + q^{-2}[0, 0]) \cdot [1] = (q^2 - q^{-2}) \cdot [0, 0, 1]
 \end{aligned}$$

Therefore, applying (2.3.3) and the relevant definitions, we deduce that

$$\begin{aligned}
 r_{[0, 0, 1]} &= \Xi(\langle [0, 0, 1] \rangle) \\
 &= \Xi([0], \langle [0, 1] \rangle)_q \\
 &= [0] * r_{[0, 1]} - r_{[0, 1]} * [0] \\
 &= (q^2 - q^{-2})([0] * [0, 1] - [0, 1] * [0]) \\
 &= (q^2 - q^{-2})^2 [0, 0, 1]
 \end{aligned}$$

Once again, using (2.2.1), we deduce that for all $k \geq 2$,

$$[0] * [0, \dots, k] - [0, \dots, k] * [0] = ([0] * [0, \dots, k - 1] - [0, \dots, k - 1] * [0]) \cdot [k]. \quad (\text{A.2.1})$$

Assume $k \geq 2$. Then, $(\alpha_0, \alpha_0 + \dots + \alpha_k) = 0$, so iterated applications of (A.2.1) yields

$$\begin{aligned}
 r_{[0,0,\dots,k]} &= [0] * r_{[0,\dots,k]} - r_{[0,\dots,k]} * [0] \\
 &= (q^2 - q^{-2})^k ([0] * [0, \dots, k] - [0, \dots, k] * [0]) \\
 &= (q^2 - q^{-2})^k ([0] * [0, 1] - [0, 1] * [0]) \cdot [2, \dots, k] \\
 &= (q^2 - q^{-2})^{k+1} [0, 0, \dots, k]
 \end{aligned}$$

Now, assume that $k \geq 2$, and $0 < j < k$. To compute $r_{[j,\dots,0,0,\dots,k]}$, we need the following. For $|j - k| > 1$,

$$\begin{aligned}
 [j] * [j - 1, \dots, k] - q^{-2} [j - 1, \dots, k] * [j] & \tag{A.2.2} \\
 = ([j] * [j - 1, \dots, k - 1] - q^{-2} [j - 1, \dots, k - 1] * [j]) \cdot [k].
 \end{aligned}$$

For $j = k - 1$,

$$\begin{aligned}
 [j] * [j - 1, \dots, 0, 0, \dots, j + 1] - q^{-2} [j - 1, \dots, 0, 0, \dots, j + 1] * [j] & \tag{A.2.3} \\
 = (q^2 [j] * [j - 1, \dots, 0, 0, \dots, j] - q^{-2} [j - 1, \dots, 0, 0, \dots, j] * [j]) \cdot [j + 1].
 \end{aligned}$$

Finally,

$$\begin{aligned}
 q^2 [j] * [j - 1, \dots, 0, 0, \dots, j] - q^{-2} [j - 1, \dots, 0, 0, \dots, j] * [j] & \tag{A.2.4} \\
 = ([j] * [j - 1, \dots, 0, 0, \dots, j - 2] - q^{-2} [j - 1, \dots, 0, 0, \dots, j - 2] * [j]) \cdot [j, j + 1].
 \end{aligned}$$

Indeed, (A.2.2) and (A.2.3) are straightforward applications of (2.2.1). Equation (A.2.4) involves a little more calculation:

$$\begin{aligned}
 & q^2 [j] * [j - 1, \dots, 0, 0, \dots, j] - q^{-2} [j - 1, \dots, 0, 0, \dots, j] * [j] \\
 &= q^2 [j - 1, \dots, 0, 0, \dots, j, j] + q^{-2} ([j] * [j - 1, \dots, 0, 0, \dots, j - 1] \\
 &\quad - [j - 1, \dots, 0, 0, \dots, j - 1] * [j]) \cdot [j] - q^{-2} [j - 1, \dots, 0, 0, \dots, j, j] \\
 &= (q^2 - q^{-2}) \cdot [j - 1, \dots, 0, 0, \dots, j, j] + q^{-2} ([j - 1, \dots, 0, 0, \dots, j] \\
 &\quad + q^2 ([j] * [j - 1, \dots, 0, 0, \dots, j - 2]) \cdot [j - 1] - ([j - 1, \dots, 0, 0, \dots, j - 2] * [j]) \cdot [j - 1] \\
 &\quad - q^4 [j - 1, \dots, 0, 0, \dots, j]) \cdot [j] \\
 &= ([j] * [j - 1, \dots, 0, 0, \dots, j - 2] - q^{-2} [j - 1, \dots, 0, 0, \dots, j - 2] * [j]) \cdot [j, j + 1],
 \end{aligned}$$

Note that (A.2.2) holds for both $[j - 1, j, \dots, k]$ and $[j - 1, \dots, 0, 0, \dots, k]$.

Now, assume that we have shown that $r_{[j-1, \dots, 0, 0, \dots, k]} = (q^2 - q^{-2})^{j+k} [j-1, \dots, 0, 0, \dots, k]$. Then, since $(\alpha_j, 2\alpha_0 + \dots + 2\alpha_{j-1} + \alpha_j + \dots + \alpha_k) = -2$,

$$\begin{aligned}
 r_{[j, \dots, 0, 0, \dots, k]} &= [j] * r_{[j-1, \dots, 0, 0, \dots, k]} - q^{-2} r_{[j-1, \dots, 0, 0, \dots, k]} * [j] \\
 &= (q^2 - q^{-2})^{j+k} [j] * [j-1, \dots, 0, 0, \dots, k] - q^{-2} [j-1, \dots, 0, 0, \dots, k] * [j] \\
 &= (q^2 - q^{-2})^{j+k} ([j] * [j-1, \dots, 0, 0, \dots, j+1] \\
 &\quad - q^{-2} [j-1, \dots, 0, 0, \dots, j+1] * [j]) \cdot [j+2, \dots, k] && \text{by (A.2.2)} \\
 &= (q^2 - q^{-2})^{j+k} (q^2 [j] * [j-1, \dots, 0, 0, \dots, j] \\
 &\quad - q^{-2} [j-1, \dots, 0, 0, \dots, j] * [j]) \cdot [j+1, \dots, k] && \text{by (A.2.3)} \\
 &= (q^2 - q^{-2})^{j+k} ([j] * [j-1, \dots, 0, 0, \dots, j-2] \\
 &\quad - q^{-2} [j-1, \dots, 0, 0, \dots, j-2] * [j]) \cdot [j, \dots, k] && \text{by (A.2.4)} \\
 &= (q^2 - q^{-2})^{j+k} ([j] * [j-1] - q^{-2} [j-1] * [j]) \cdot [j-2, \dots, 0, 0, \dots, k] && \text{by (A.2.2)} \\
 &= (q^2 - q^{-2})^{j+k+1} [j, \dots, 0, 0, \dots, k].
 \end{aligned}$$

Finally, one computes using (2.4.7) and (2.2.6) that the coefficient of $[j, \dots, 0, 0, \dots, k]$ in (2.4.9) is

$$\frac{(-1)^{j+k} (q^2 - q^{-2})^{j+k+1} (1 - q^4)}{q^{-2(j+k)} (1 - q^2)^2 (1 - q^4)^{j+k}} = [2]_0^2,$$

so the result follows. ■

Proposition A.2.3. *In type C_r :*

$$\begin{aligned}
 b^*[i, \dots, j] &= [i, \dots, j], \quad 0 \leq i \leq j < r, \\
 b^*[j, \dots, 1, 0, 1, \dots, k] &= [j, \dots, 1, 0, 1, \dots, k], \quad 1 \leq j < k < r, \\
 b^*[0, \dots, j, 1, \dots, j] &= q[0] \cdot ([1, \dots, j] * [1, \dots, j]), \quad 1 \leq j < r.
 \end{aligned}$$

Proof: The first two formulae can be proved by induction as in the type A case.

We now prove that $b^*[0, \dots, j, 1, \dots, j] = q[0] \cdot ([1, \dots, j] * [1, \dots, j])$. Our argument is essentially the same as [Le, Lemma 53]. Indeed, $[1, \dots, j]$ belongs to \mathcal{U}_q , so $[1, \dots, j] * [1, \dots, j]$ belongs to \mathcal{U}_q . Using [Le, Theorem 5], we deduce that $f = [0] \cdot ([1, \dots, j] * [1, \dots, j])$ belongs to \mathcal{U}_q . Clearly $\min(f) = [0, \dots, j, 1, \dots, j]$ so by Theorem 2.4.2(i) f is proportional to $b^*[0, \dots, j, 1, \dots, j]$. Finally, using Theorem 2.4.2(i) and (2.4.3) with $g = [1, \dots, j, 1, \dots, j]$ we obtain the result. ■

Proposition A.2.4. *In type D_r :*

$$\begin{aligned}
 b^*[0] &= [0] \\
 b^*[0, 2, \dots, i] &= [0, 2, \dots, i], \quad 2 \leq i < r, \\
 b^*[i, \dots, j] &= [i, \dots, j], \quad 1 \leq i \leq j < r, \\
 b^*_{[0, \dots, j]} &= [1, 0, 2, \dots, j] + [0, 1, 2, \dots, j], \quad 2 \leq j < r, \\
 b^*[j, \dots, 2, 1, 0, 2, \dots, k] &= [j, \dots, 2, 1, 0, 2, \dots, k] + [j, \dots, 2, 0, 1, 2, \dots, k], \quad 2 \leq j < k < r.
 \end{aligned}$$

Proof: All cases follow by an easy induction argument that we leave as an exercise for the reader. ■

A.3. **The F_4 case.** Calculations available upon request.

A.4. **The E_8 case.**

Height	Good Lyndon Words
1	[0], [1], [2], [3], [4], [5], [6], [7]
2	[01], [13], [23], [34], [45], [56], [67]
3	[013], [123], [134], [234], [345], [456], [567]
4	[0123], [2134], [2345], [0234], [1345], [3456], [4567]
5	[02134], [21345], [13456], [32134], [02345], [23456], [34567]
6	[021345], [213456], [302134], [321345], [023456], [134567], [234567]
7	[2134567], [2302134], [4321345], [0213456], [0234567], [3213456]
8	[02134567], [23021345], [43012345], [32134567], [30213456], [43213456]
9	[423021345], [230213456], [43213456], [543213456], [430213456], [423021345], [302134567]
10	[3423021345], [4230213456], [5430213456], [5432134567], [2302134567], [4302134567]
11	[13423021345], [54230213456], [34230213456], [42302134567], [54302134567], [5432134567]
12	[534230213456], [134230213456], [342302134567], [54230213456], [654302134567]
13	[4534230213456], [5134230213456], [1342302134567], [5342302134567], [6542302134567]
14	[45134230213456], [51342302134567], [45342302134567], [65342302134567]
15	[314534230213456], [451342302134567], [651342302134567], [645342302134567]
16	[2314534230213456], [3145342302134567], [6451342302134567], [5645342302134567]
17	[02314534230213456], [23145342302134567], [63145342302134567], [56451342302134567]
18	[023145342302134567], [623145342302134567], [563145342302134567]
19	[6023145342302134567], [562345342302134567], [4563145342302134567]
20	[56023145342302134567], [4562345342302134567]
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23	[13456023145342302134567], [23456023145342302134567]
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25	[3213456023145342302134567]
26	[43213456023145342302134567]
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